Creating the Perfect Balance

Symmetry in maths

Everything is mathematical

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ἀγεωμὲτρητος
μηδεὶς
εἰσίτω

Let no one ignorant of geometry enter here.

Inscription at the entrance to the
Academy of Plato

Preface

Tiger, tiger, burning bright
In the forests of the night,
What immortal hand or eye
Could frame thy fearful symmetry?
William Blake (1757–1827)

In many English-speaking countries, the above quotation has become part of what could be called the general culture. The "fearful symmetry" mentioned in the poem becomes clear when one looks into the face of a tiger, though no one coming unexpectedly face to face with one would stop to reflect on whether the face is symmetrical or not. However, a tiger's head has an obvious symmetry. But what is it that we describe as 'symmetrical'? Are widespread religious symbols, such as the triquetra, a continuous set of three arcs that form a triangular knot (both a sign of the Holy Trinity of Christianity and present in Germanic pagan writings), or the mystical *Tetractys* figure (a triangle of triangles) of the Pythagoreans, more symmetrical than a simple polyhedron?

Leaving aside any non-earthly properties, the equilateral triangle has a symmetry group of order 6; in other words, it has six elements. A simple star polyhedron can have a symmetry group with an order of tens or hundreds; in other words, the shape has tens or hundreds of elements. A simple, utterly empty circle has still more symmetries: an infinite number. But may our souls take comfort: there are still some symmetrical triangular objects that are less symmetrical than the symbol of the Holy Trinity or the *Tetractys*. The centre of the coat of arms of the Isle of Man (see page 41), which dates from 1266, shows a *triskelion* or three-legged figure that has a symmetry group of only three elements, fewer elements than the triquetra's group.

The triangle group is finite, but it is dihedral with axial rotations and symmetries. On the other hand, the Isle of Man's group, though also finite, belongs to the groups called 'cyclic groups' which have no axial symmetries. It is also a subgroup that is isomorphic to a subgroup of the triquetra's group. And, we can go further by adding that the Manx group of the rotating legs is also isomorphic to a subgroup

– a normal subgroup, to boot – of the group formed by the hours of a clock. This group – both cyclic and Abelian – is of 12 elements, as everyone knows.

Fear not if you are lost at this point! Almost all of the above will be within your grasp by the end of this book, but we may be left with a larger hole in our knowledge. But does everyone really know what is most or least symmetrical? To better understand the symmetry of the world that surrounds us requires some basic mathematical tools, such as the concept of groups. Funnily enough, this field of maths was not invented with the aim of getting a better understanding of the symmetries, but that is what it is used for today. The concept of groups is an algebraic device, already sensed by luminaries such as Lagrange and later developed in masterly fashion by Galois for taking on the challenge of solving polynomial equations without considering their geometric aspects. The fruits of their labours, group theory – the modern gateway to symmetry – is neither straightforward nor easy. It cannot be learned by just reading a single book on mathematics, but it is an awesomely beautiful theory.

Galois died at the age of 20 after a pistol duel over a spat about politics, women – and maths. Shortly before he died, the hyperactive Galois wrote his theories in the hope that they would survive and reach the hands of a more celebrated mathematician. They only had a decade to wait. The story provides as much material for a novel as it does for a book on algebra – the same could be said about the 16th century dispute between Niccolò Tartaglia and Gerolama Cardano over polynomial equations. And what about the longest theorem ever written, the classification of the so-called "simple groups"? It's so long that no single person has read all of it.

Paradoxically, we shall hardly touch on the phenomenon referred to by the words "through the looking glass" which appear on the cover of the second *Alice* novel written by Lewis Carroll, and which, perhaps, represents the full extent of the interest that most people have in what they understand as symmetry.

Symmetry is the shoes that are not the same, even though they are from the same pair (experts say they are *enantiomorphic*); it is a palindromic sentence; it is found in the human body's proteins (*laevorotatory*); and the reversibility of the laws of physics. But the symmetries right/left – or axial – are elementary aspects of authentic symmetry. There is already sufficient literature on that issue, and we shall not be giving you just another book on the topic of right and left. There is a great deal more of symmetry and groups awaiting the spotlight.

There are finite groups and infinite groups, and all play a vital role in describing and understanding the macroscopic world, and the microscopic one, too. On the

small scale, there are not only the radiolarians, diatoms and viruses that show external symmetry, but crystals with their internal symmetries, too. The Universe itself is a box of quantum symmetries, as physicists long ago discovered. And they still do not know all of them; perhaps when they do we shall come across the longed-for 'theory of everything' that humanity has been seeking for centuries. Nevertheless, symmetries (and their violations) do not explain everything, but they are waiting there for us to understand them. So let's begin.

Chapter 1

What is Symmetry?

I just don't understand; why is this symmetry stuff so important?
Mao Tse-tung

In mathematics, the term 'symmetry' does not refer to exactly the same thing as in other sciences, or more to the point, daily life. Mathematicians are very precise, though perhaps punctilious or obsessive would be better descriptions, and by symmetry they understand something very particular and precise. For them, anything that differs from their definition is not symmetrical in the strict sense of the word. There is a joke that portrays mathematical intransigence perfectly. An astronomer, an engineer and a mathematician are travelling together through Scotland on a train. The astronomer looks out of the window and is surprised and delighted to see a black sheep prancing about in the field alongside the carriage. Perhaps a little over hastily, he remarks to his travelling companions, "How curious. In Scotland all the sheep are black." The engineer kindly corrects him: "No, no. It's just that some sheep are black." And then the mathematician self-importantly chips in with, "The actual fact of the matter is that in Scotland there is at least one field, which contains at least one sheep, one of whose sides is black." Something very similar happens with symmetry and mathematics. When everybody professes to know what a certain thing is, it becomes very difficult to define.

The concept of symmetry

For the moment we shall just stay with the loosest, vaguest, least abstract or demanding concept of symmetry, and we'll just stick to exploring within the boundaries of the world that surrounds us, which we'll understand as being equipped with all the instruments and characteristics of Euclidean geometry.

We shall begin with objects. The term 'symmetrical' describes an object that, on being moved, coincides with itself without deformation in a 'normal' manner. In other words, we are going to limit ourselves to a concept of symmetry that maintains all distances

and lengths unaltered. For readers who are well informed in advanced mathematics, we can say that for the moment we are going to limit the concept of symmetry to what is called isometries (from the Greek *isos* = equal, and *metros* = measurement).

Let's first deal with one-dimensional objects. A segment (of a line), for example, is symmetrical in the most trivial of ways:

$$a \;\text{———————}\; b \quad b \;\text{———————}\; a$$

but there is only one way to move it so that it ends up as it was. If the points are interchanged, which in our three-dimensional space is equivalent to turning it 180° around the central point, the segment remains invariant. The term *invariant* means that if the segment were not accompanied by those letters, the segment that results from the rotation would be indistinguishable from the original. In actual fact, an exacting mathematician would say that there are two movements that leave the segment invariant, the aforementioned rotation of 180° and a 360° rotation, or plain rotation. This latter rotation is, in all effects, identical to not moving it (0°) or moving it 720° (rotating it twice), or 1,080° (three rotations) – we needn't go on.

In algebra, a segment is said to be symmetrical and its group of movements is formed by the identity (equivalent to $0° = 0$ radians, $360° = 2\pi$ radians, $720° = 4\pi$ radians, etc.) and the rotation of 180° ($180° = \pi$ radians, $540° = 3\pi$, $900° = 5\pi$ radians, etc.). Note that the degrees have their equivalents in radians. This is more important than it may seem. Angles have been measured in radians since infinitesimal calculus appeared more than 300 years ago.

It seems that working in one dimension does not give us much scope. But when you think about it a bit more, there is, in fact, a one-dimensional object that gives more symmetry. A straight line itself

$$A$$

———————————————————————

gives more, even though it doesn't seem like it; any translation to the right or the left, and of any length, leaves the straight line as it was – invariant. There is an infinite number of translations of the straight line, such as we have defined symmetries. That is, if we give the straight line a point of origin (point zero) and we identify the line's points with the set of real numbers $\mathbb{R}$, the translation t

$$x \rightarrow x + t$$

RADIANS AND DEGREES

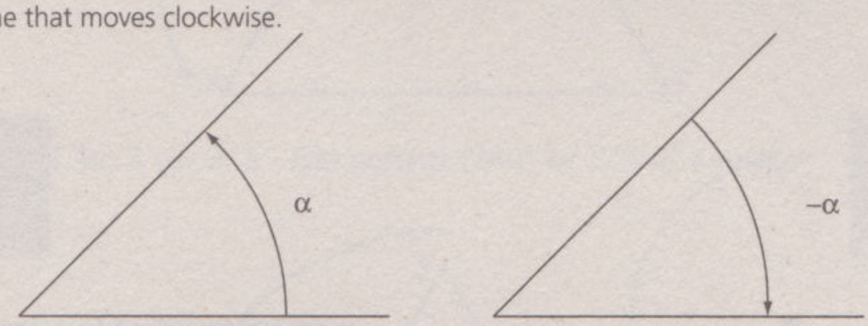
The equivalence between sexagesimal (base 60) degrees and radians is provided by the definition of radian itself: an angle that gives an arc that is equal to the radius of the circumference used to measure it. In other words:

$$1 \text{ radian} \approx 57°17'45''$$
$$\pi \text{ radians} = 180°.$$

Degrees	0°	30°	45°	60°	90°	180°	270°	360°
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π

In this book, radian notation will be used. The use of sexagesimal degrees is still very widespread (particularly in elementary geometry), by tradition, for convenience (no fractions are needed), and, it has to be said, through inertia – and due to the influence of manufacturers of optical instruments. However the international system of measurements has the radian as the standard measurement of angles, not the degree.

Lastly, let's agree on a simple rule. If we ever need to consider that the orientation of an angle is important, we'll understand anti-clockwise as being the positive direction. A negative angle will be one that moves clockwise.



Direction of an angle.

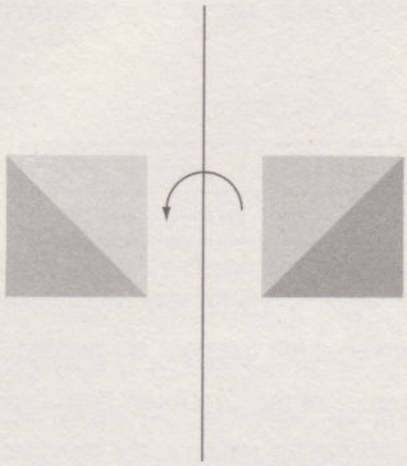
The radian, a much more natural way of measuring angles than the degree, was introduced in 1713 by Roger Cotes (1682–1716), an ill-fated English mathematician and friend of Newton.

which transforms each point x into $x + t$ is a symmetry of straight line A, as it translates the line over itself by moving all of its points the amount t to one side or another (and although we won't go into it here, there are still more symmetries in $\mathbb{R}$).

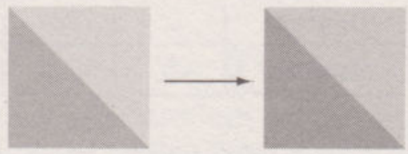
Later on, when the reader knows more about symmetry, he or she will see that this translation is an element of the Lie group called $GL(1, \mathbb{R})$..., but let's not get ahead of ourselves.

One more dimension

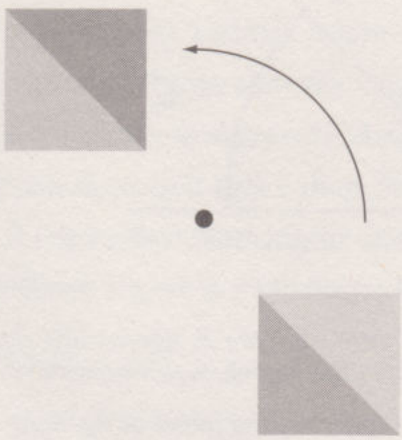
Let's go one step further by adding a dimension and moving on to two-dimensional figures. On a plane, we can find a few examples of symmetrical movements that maintain distances, as can be seen in the figures below.



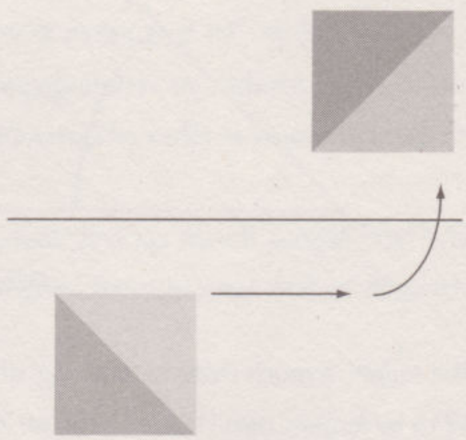
Reflection



Translation

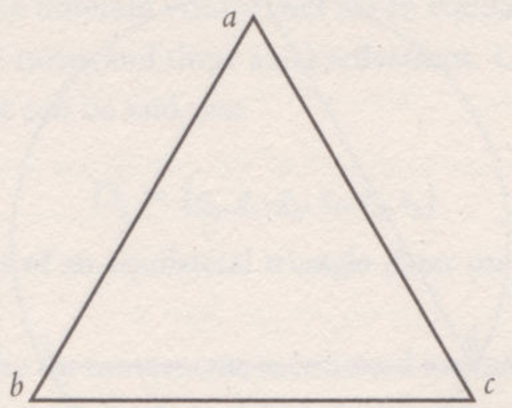


Rotation

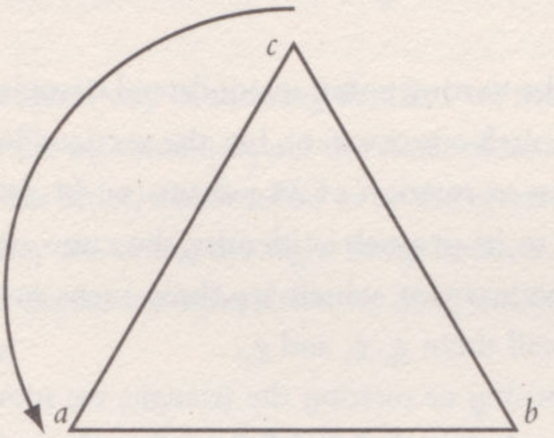


Reflection with slide

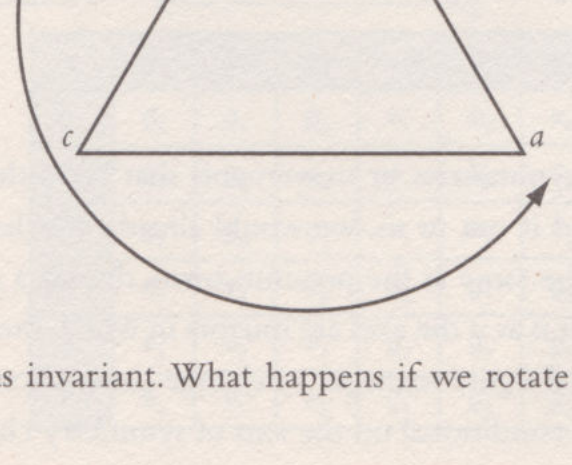
Let's apply it now to something as elementary as a triangle of vertices *a*, *b* and *c*. Take a look at the symmetry of an equilateral triangle, a perfect polygon with vertices *a*, *b* and *c* and with a central point, which is the circumcentre, barycentre and incentre of the triangle all at the same time.



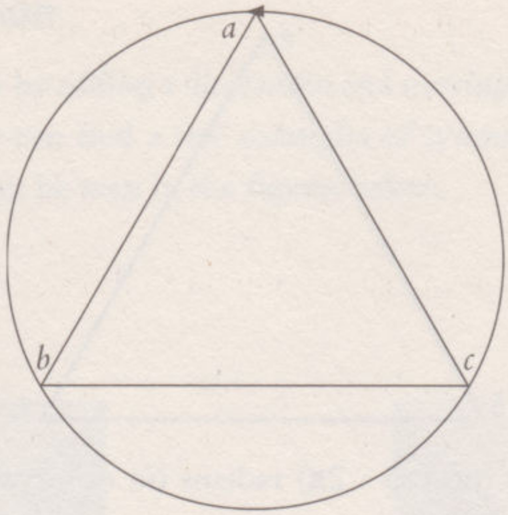
If it is rotated $2\pi/3$ (or $1/3 \cdot 2\pi$) radians (in other words, 120° or 1/3 of a complete turn) around the centre,



or it is rotated $4\pi/3$ (or $2/3 \cdot 2\pi$) radians (240° or 2/3 of a turn),

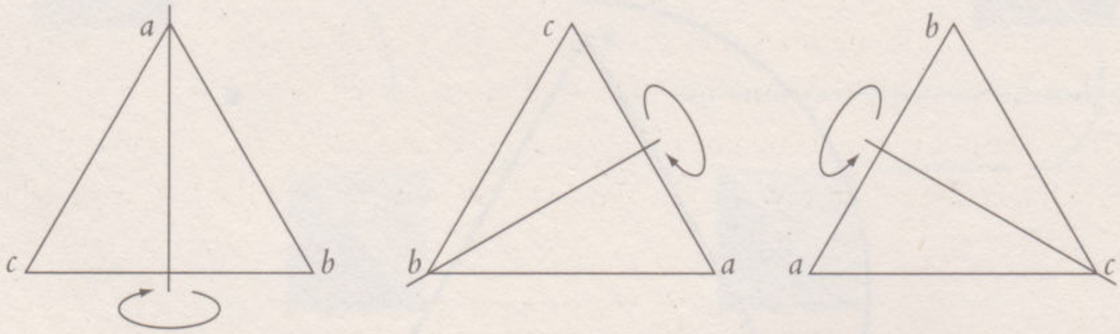


the triangle remains invariant. What happens if we rotate it 2π radians (360°; a complete rotation)?



Then we return to the starting point; an equilateral triangle is not only invariant when it is subjected to such a movement, but the vertices' letters are the same as at the beginning. A turn or rotation of 2π radians (or 4π, 6π, 8π, etc.) is a trivial turn or rotation, equal to its geometric identity, the same as not moving it at all. We have, then, three symmetries, which are three turns or rotations of 0, $2\pi/3$ and $4\pi/3$ radians; let's call them g_0, g_1 and g_2.

If now, instead of turning or rotating the triangle, we submit it to a reflection, taking as its axis the straight line that passes through *a*, through *b* or through *c*,



we get another three symmetries or movements that leave the triangle invariant. Even if no one pointed it out to us, we would already see that these symmetries, apart from not being the same as the previous turns, display a characteristic which specialists call torsion. It is as if the axes are mirrors in which the triangle is reflected, with which the original figure manages to change orientation. We shall call these symmetries, which are conditional on the axis of symmetry chosen, e_1, e_2, e_3.

In this way we get six different symmetries for an equilateral triangle: $g_0, g_1, g_2, e_1, e_2, e_3$, three rotations or turns, and three axial reflections. Using the modern group theory nomenclature, it can be said that

$$D_3 = \{g_0, g_1, g_2, e_1, e_2, e_3\}$$

is the set of symmetries of an equilateral triangle (later on we'll explain why it has the name D_3).

When any two of the six movements mentioned are made, let's call them m_i and m_j, the result is another movement, m_k, which is also a movement pertaining to D_3.

In the expression

$$m_j \bullet m_i = m_k$$

the sign • is an abbreviated way of showing that first we move the triangle as shown by m_i, then we make the movement m_j and the result obtained is movement m_k. Professionals write it in this order, which seems to be the reverse of what common sense would dictate, but they are doing the right thing because it makes things easier at more advanced levels. So, for example, if first we do e_3 and then g_1, it will be expressed in the following way:

$$g_1 \bullet e_3 = e_2.$$

Mathematicians sum up all the above in just a few words by stating that • is an operation of D_3. Sometimes they say that D_3 is a 'closed set' for the operation •, which means exactly the same but expressed in a more complicated way.

If we set out the movements of D_3 in table form, in rows and columns, in such a way that in the cell in row n and column p is $m_n \bullet m_p$, we get the following table:

•	g_0	g_1	g_2	e_1	e_2	e_3
g_0	g_0	g_1	g_2	e_1	e_2	e_3
g_1	g_1	g_2	g_0	e_3	e_1	e_2
g_2	g_2	g_0	g_1	e_2	e_3	e_1
e_1	e_1	e_2	e_3	g_0	g_1	g_2
e_2	e_2	e_3	e_1	g_2	g_0	g_1
e_3	e_3	e_1	e_2	g_1	g_2	g_0

We have designed the rotations beginning with the subscript 0 because g_0 is rather a special rotation, a movement of the triangle that is different from the others. Let's see why.

For all symmetry or movement m of the equilateral triangle, whatever it is, it verifies

$$m \bullet g_0 = g_0 \bullet m = m.$$

It is as if the movement g_0 behaved like a neutral movement that changes nothing, neither operating before or after, neither to the right nor to the left. We can check on this by looking at the previous table: the rows and columns corresponding to g_0 are left unaltered. Such an element behaving in this way in an operation is called a neutral element.

In basic arithmetic, we already have some very common examples of a neutral element. If the operation is addition:

$$m + 0 = 0 + m = m$$

whatever the number m is, and 0 is the neutral element for operation + in the world of arithmetic.

Analogously, and if we take the multiplication operation,

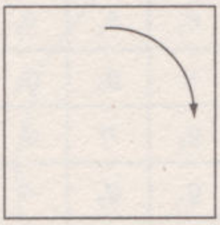
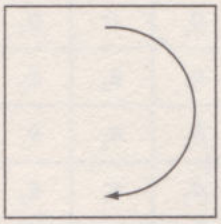
$$m \cdot 1 = 1 \cdot m = m$$

and 1 turns out to be the neutral element in operation · among the numbers.

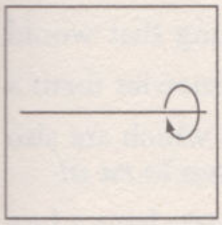
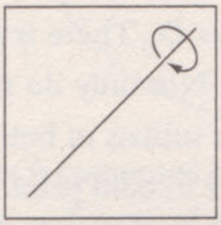
By going back a few lines, let's change the notation g_0 to n for *neutral*. Therefore, we'll now write the table of operation • like this:

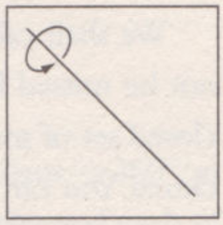
•	n	g_1	g_2	e_1	e_2	e_3
n	n	g_1	g_2	e_1	e_2	e_3
g_1	g_1	g_2	n	e_3	e_1	e_2
g_2	g_2	n	g_1	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

That is the table for set D_3. It is the table of the symmetries of the equilateral triangle. What would happen in the case of a square, which is clearly more symmetrical than a triangle? A square has in fact eight elemental symmetries, as can be seen in the diagrams below,


n g_1 g_2 g_3



e_1 e_2 e_3 e_4

and four of them change the direction of reading, they present torsion.

This stamp clearly demonstrates the elemental symmetries of a square.

A square's symmetries can also be subjected to the operation • and give the following table:

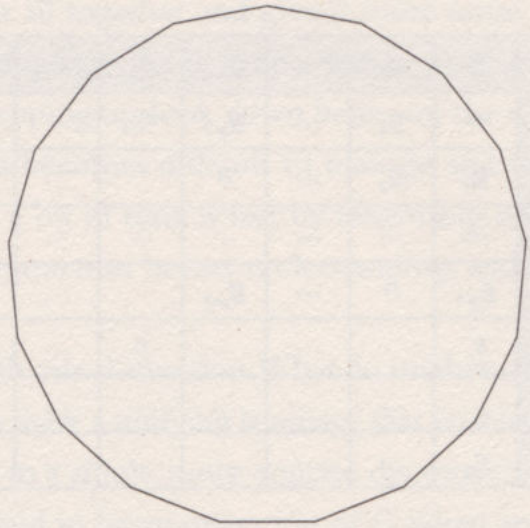
•	n	g_1	g_2	g_3	e_1	e_2	e_3	e_4
n	n	g_1	g_2	g_3	e_1	e_2	e_3	e_4
g_1	g_1	g_2	g_3	n	e_4	e_3	e_1	e_2
g_2	g_2	g_3	n	g_1	e_2	e_1	e_4	e_3
g_3	g_3	n	g_1	g_2	e_3	e_4	e_2	e_1
e_1	e_1	e_4	e_2	e_3	n	g_2	g_3	g_1
e_2	e_2	e_3	e_1	e_4	g_2	n	g_1	g_3
e_3	e_3	e_1	e_4	e_2	g_1	g_3	n	g_2
e_4	e_4	e_2	e_3	e_1	g_3	g_1	g_2	n

We shall call the square's set of symmetries D_4. There is something that would not be missed by the keen eye of a specialist. Not only do the symmetries form a closed set of movements, but here are parts or subsets of both tables which are also closed. You can see this by looking at the parts in bold print.

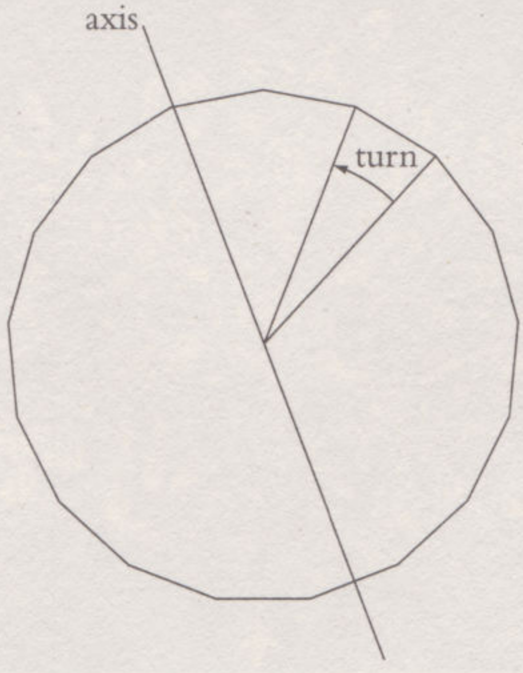
•	$\mathbf{n}$	$\mathbf{g_1}$	$\mathbf{g_2}$	e_1	e_2	e_3
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{g_1}$	$\mathbf{g_2}$	e_1	e_2	e_3
$\mathbf{g_1}$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{n}$	e_3	e_1	e_2
$\mathbf{g_2}$	$\mathbf{g_2}$	$\mathbf{n}$	$\mathbf{g_1}$	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

•	$\mathbf{n}$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	e_1	e_2	e_3	e_4
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	e_1	e_2	e_3	e_4
$\mathbf{g_1}$	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{n}$	e_4	e_3	e_1	e_2
$\mathbf{g_2}$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{n}$	$\mathbf{g_1}$	e_2	e_1	e_4	e_3
$\mathbf{g_3}$	$\mathbf{g_3}$	$\mathbf{n}$	$\mathbf{g_1}$	$\mathbf{g_2}$	e_3	e_4	e_2	e_1
e_1	e_1	e_4	e_2	e_3	n	g_2	g_3	g_1
e_2	e_2	e_3	e_1	e_4	g_2	n	g_1	g_3
e_3	e_3	e_1	e_4	e_2	g_1	g_3	n	g_2
e_4	e_4	e_2	e_3	e_1	g_3	g_1	g_2	n

Let's just skip forward now and take a look at the symmetries of a regular convex polygon with p sides. By using a bit of imagination, we'll suppose that a polygon such as the one below has p sides (in actual fact it has 17).



Its set of symmetries is composed of p rotations, each one $2\pi/p$ radians ($\approx 360°/p$) and p axial symmetries with axes, of which $p/2$ pass through the centre and each of the polygon's vertices and the others $p/2$ pass through the centre and the median points of the sides. In all, $2p$: p turns and p axial symmetries.



By calling the rotations g_i (except for the neutral rotation, which we'll continue to call n) and the axial symmetries e_i, we obtain a set of $2p$ elements, which we'll call D_p.

$$D_p = \{n, g_1 \dots, g_{p-1}, e_1 \dots, e_p\}.$$

We won't write its table because it can't be done (p is not a fixed number). However, at least one closed subset can be marked in it for the operation •, although there are a lot more:

•	n	g_1	...	g_{p-1}	e_1	...	e_p
n	n	g_1	...	g_{p-1}	e_1	...	e_p
g_1	g_1	g_2	...	n		...	
...	...	...	...	...	...	...	...
g_{p-1}	g_{p-1}	n	...	g_{p-2}		...	
e_1	e_1		...		n	...	...
...	...	...	...	...	...	...	...
e_p	e_p		...		...	...	n

There are a lot of objects related to regular polygons and dihedral groups in nature. One of the best known is a snowflake, which shows symmetry group D_6.

Photo of a snowflake (taken in the early 20th century by Wilson Bentley). Ice crystals have a hexagonal structure and belong to symmetry group D_6.

An abstract interlude

We could go on – and in fact we will do so later – by analysing symmetry sets, but at this point, perhaps we should pause and put everything in order. To help us to interpret it all, to put it all together and give it more sense, we can make a simple change in the type of language, going from a purely descriptive scenario to a more abstract level, and by stepping up, look down better on the whole scene. That's what scientists do when data becomes difficult to manage and they can't see the wood for the trees. Although a bit of time is lost by refocusing on things and going over them, time spent on abstraction boosts understanding, and in the end it turns out to be worth the effort.

Let's go on then with our abstraction. What do mathematicians take from everything said up to now? Is there a unifying language that enables all the aforementioned maths to be integrated in a single, more concise discourse?

Given a set G, $\bullet$ is said to be an operation of G when any two elements of G, a and b, always determine a third c, which is expressed as follows:

$$a \bullet b = c.$$

So, c belongs to G. In set theory, this is shown by the expression $c \in G$. The operation $\bullet$ is an internal operation, or, in other words, a closed operation; it is a G operation.

G is said to be a group when the operation $\bullet$ and G fulfil three conditions:

1. There is an element n in G (n is called a neutral element of G) that verifies, for every element $g \in G$

$$g \bullet n = n \bullet g = g.$$

2. For every $g \in G$ there is another element, which we'll call the inverse of g and we'll call it g^{-1}, so that

$$g \bullet g^{-1} = g^{-1} \bullet g = n.$$

3. All the elements of G, for example $a, b, c \in G$, satisfy the associative property

$$(a \bullet b) \bullet c = a \bullet (b \bullet c).$$

All the sets of symmetry elements enumerated up to now fulfil those three conditions. In this way, we can refer to the line group (which has 2 elements), the equilateral triangle group (which has 6 elements), the square group (which has 8 elements) and to the group of regular polygons with n sides (which has $2n$ elements). The number of elements of a G group, which is denoted $|G|$, is called the order of G.

Groups with a finite number of elements are known as finite groups, and those which have infinite elements are called infinite groups. The set of translations of the straight line A is an infinite group. The set of integers

$$\mathbb{Z} = \{\ldots, -3. -2, -1, 0, 1, 2, 3\ldots\}$$

is also an infinite group with the operation $+$. The neutral element is 0 and the inverse element of any integer p is the integer $-p$. When the operation is addition, the inverse element is normally denoted as opposed

$$p + (-p) = (-p) + p = 0,$$

though, to abbreviate, we usually write

$$p - p = -p + p = 0.$$

The set $\mathbb{R}*$ of non-null real numbers (that is, $\mathbb{R}$ without the zero) is also a group under the operation $\cdot$ or multiplication. The inverse of any number r is $1/r$ and the neutral element of the multiplication $\cdot$ is 1.

$$r \cdot 1/r = 1/r \cdot r = 1.$$

A subset or F part of G, $F \subset G$, which is also a group under operation $\bullet$ (but of a lesser order), is called a subgroup. In some previous examples we have highlighted some clear subgroups in bold print (in both cases these were the rotations).

A fundamental result that is valid for finite groups is the Lagrange theorem. It's rather complicated to demonstrate this theorem (and it will not be reproduced here), but the formulation is simple enough: If F is a subgroup of G, the order of F or number of elements of F – which we call $|F|$ – divides the order of G, which previously we denominated $|G|$. This result, which limits the possible subgroups to sets with a determined number $|F|$ of elements that necessarily has to be a divisor

of $|G|$, bears Lagrange's name, but it is not entirely down to him; he only partially proved it. The theorem was finally proved by the Italian, Pietro Abbati (1768–1842).

Let's take, for example, D_3, which verifies $|D_3| = 6$, as it has 6 elements. So, it can only have subgroups of 6, 3, 2 or 1 elements, as the order of a subgroup has to be a divisor of 6. Its corresponding tables are given below with subgroups in bold:

•	**n**	**g_1**	**g_2**	**e_1**	**e_2**	**e_3**
n	**n**	**g_1**	**g_2**	**e_1**	**e_2**	**e_3**
g_1	**g_1**	**g_2**	**n**	**e_3**	**e_1**	**e_2**
g_2	**g_2**	**n**	**g_1**	**e_2**	**e_3**	**e_1**
e_1	**e_1**	**e_2**	**e_3**	**n**	**g_1**	**g_2**
e_2	**e_2**	**e_3**	**e_1**	**g_2**	**n**	**g_1**
e_3	**e_3**	**e_1**	**e_2**	**g_1**	**g_2**	**n**

Order 6

•	**n**	**g_1**	**g_2**	e_1	e_2	e_3
n	**n**	**g_1**	**g_2**	e_1	e_2	e_3
g_1	**g_1**	**g_2**	**n**	e_3	e_1	e_2
g_2	**g_2**	**n**	**g_1**	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

Order 3

•	**n**	g_1	g_2	**e_1**	e_2	e_3
n	**n**	g_1	g_2	**e_1**	e_2	e_3
g_1	g_1	g_2	n	e_3	e_1	e_2
g_2	g_2	n	g_1	e_2	e_3	e_1
e_1	**e_1**	e_2	e_3	**n**	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

Order 2

•	**n**	g_1	g_2	e_1	**e_2**	e_3
n	**n**	g_1	g_2	e_1	**e_2**	e_3
g_1	g_1	g_2	n	e_3	e_1	e_2
g_2	g_2	n	g_1	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	**e_2**	e_3	e_1	g_2	**n**	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

Order 2

•	**n**	g_1	g_2	e_1	e_2	**e_3**
n	**n**	g_1	g_2	e_1	e_2	**e_3**
g_1	g_1	g_2	n	e_3	e_1	e_2
g_2	g_2	n	g_1	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	**e_3**	e_1	e_2	g_1	g_2	**n**

Order 2

•	**n**	g_1	g_2	e_1	e_2	e_3
n	**n**	g_1	g_2	e_1	e_2	e_3
g_1	g_1	g_2	n	e_3	e_1	e_2
g_2	g_2	n	g_1	e_2	e_3	e_1
e_1	e_1	e_2	e_3	n	g_1	g_2
e_2	e_2	e_3	e_1	g_2	n	g_1
e_3	e_3	e_1	e_2	g_1	g_2	n

Order 1

The first and last tables are not of interest, as they are G itself and the trivial group $\{n\}$ which is composed only of the neutral element. The others are of order 3 (the rotations)

$$\{n, g_1, g_2\}$$

and of order 2

$$\{n, e_1\}, \{n, e_2\}, \{n, e_3\}.$$

There are no more subgroups. It's not necessary to look for possible subgroups of 4 or 5 elements as Lagrange's theorem shows that there aren't any.

Shapes that have a centre and have a finite symmetry group formed by single rotations (so-called cyclic groups, as we shall see later on) or by rotations and symmetries (dihedral groups) form a symmetry point group. Such groups are also known as Leonardo groups after the many chapels that Leonardo da Vinci designed by using these very symmetries.

When a group verifies the commutative law, that is to say, when for every $a, b \in G$

$$a \bullet b = b \bullet a$$

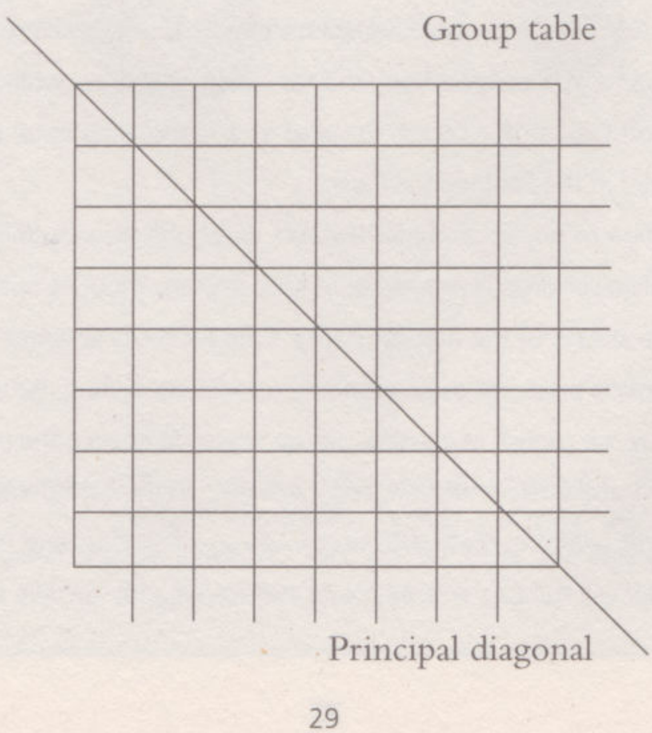
the group is said to be commutative. It is not a feature of symmetry groups to be commutative; a clear example is the triangle, as

$$e_1 \bullet g_2 = e_3$$
$$g_2 \bullet e_1 = e_2,$$

with which no commutativity is accomplished.

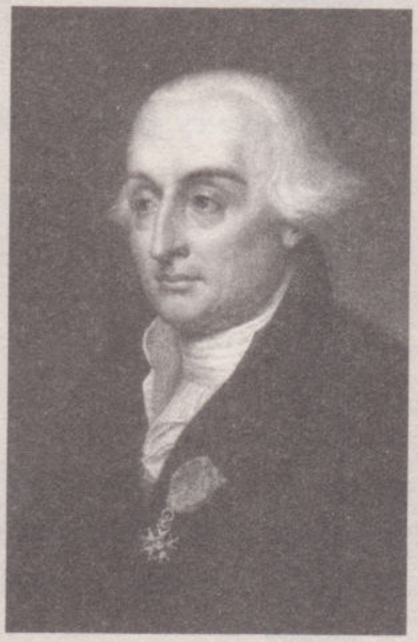
Commutative groups are also known as Abelian groups after Niels Abel (1802–1829), the Norwegian mathematician who also lends his name to the Abel prize in mathematics – an equivalent to a Nobel prize – awarded by the Norwegian Science Academy since 2003.

If the group's table is symmetrical with respect to the principal diagonal, as shown in the following diagram, the group is Abelian.



By following that rule, D_n is not commutative or Abelian for $n > 2$. The attentive reader will have noticed that we have not defined D_2, which corresponds, in all logic, to the symmetry group of the digon, or two-sided polygon. Algebraists, on account of the nature of their science, use D_2 to denote the group of double the elements

JOSEPH-LOUIS LAGRANGE (1736–1813)



Born into a military family in Turin, Lagrange soon became renowned in scientific circles thanks to his hard work and to the lessons of Leonhard Euler, whom he always revered as a teacher even though Euler never directly taught him. When Euler left his position at the Berlin Academy, Lagrange took his place, perhaps influenced by the reputation of King Frederick II, an erudite and cultured monarch, who was then known as a protector of the arts and sciences. Lagrange spent 20 years in Berlin, but after the death of the king he moved to France and was made a member of the Paris Science Academy by Louis XVI. He continued with his brilliant career, got married for a second time, this time to a considerably younger and adoring woman, and began to emerge from a deep bout of depression. But just when everything seemed to be going well for him, the French Revolution broke out in 1792, and Lagrange feared for his future and for his neck. However, his fame and discretion preserved his life and his work. He was appointed president of the Weights and Measurements Committee and devoted all his energy to establishing the metric system. He never had any problems at a political level and his status only grew with the arrival of Emperor Napoleon, who made Lagrange a count. Revered as a glorious national figure and laden with honours, he is interred in the Pantheon in Paris.

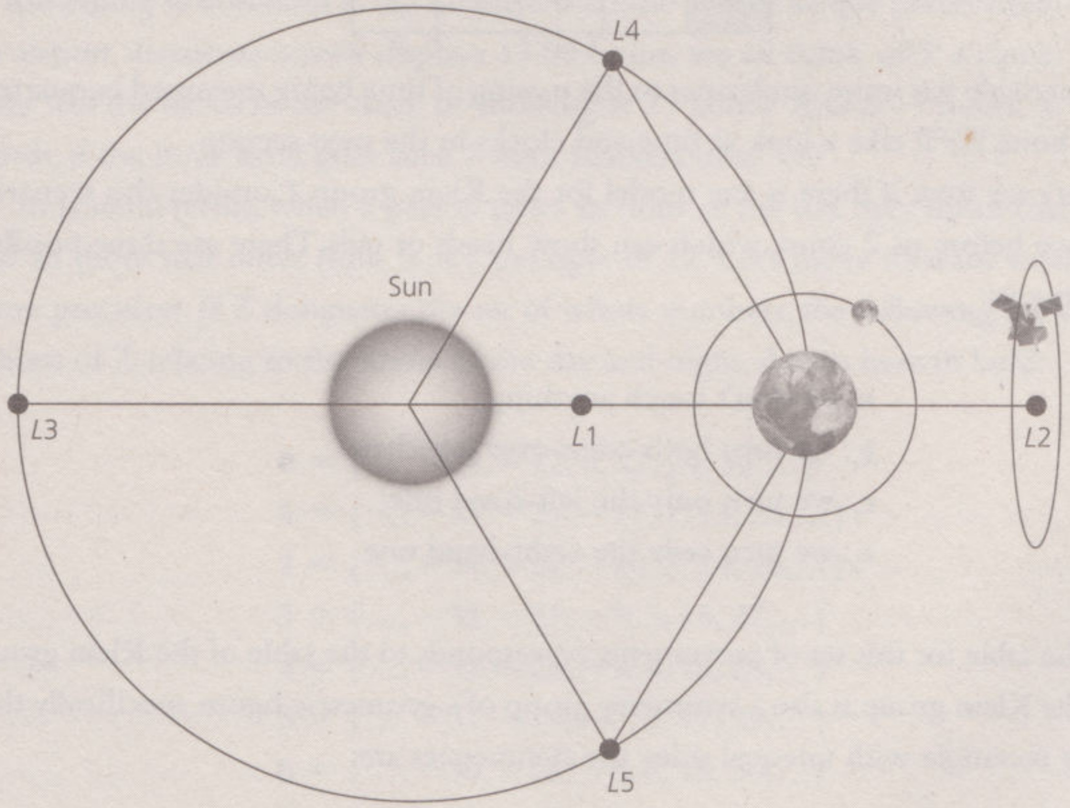
Lagrange's contribution to physics and mathematics was huge. Responsible for countless results, Lagrange was a prolific yet orderly researcher, whose articles, thought out and perfected before being published, are always of the highest clarity. Even his most abstruse texts are said to have been sent to the printing press just as he wrote them, with no editing required.

Lagrange is known as the creator of classical mechanics as he applied the principles of variational calculus to mechanics and intelligently reduced mechanics to the consideration of differential and integral equations. His contribution to astronomy was also first class and he is rightly famous for finding what we now call the Lagrangian points. All in all, Lagrange, like a modern King Midas,

of the subindex ($2 \cdot 2 = 4$) which has some rotations and reflection as elements,

$$\{n, g_1, e_1, e_2\}$$

can be said to have converted all the complicated and unapproachable subjects that he touched in the field of mechanics into polished questions that could be treated by mathematical analysis. His contributions to the theory of numbers and probability are equally outstanding, but where Lagrange changed the game was in the development of algebra and the birth of group theory, and that's why he is of particular interest to us. He dealt with such concepts within the framework of his studies on polynomial equations and in many cases provided a foretaste of the theories of another genius to come, Évariste Galois.



Lagrangian points (also called libration points), marked as L in the graphic, are situated between two heavenly bodies (such as the Earth and the Sun) in such a position that bodies located in those points are at a gravitational equilibrium. They are, therefore, the ideal places for locating geostationary satellites.

with the following table of operations:

$\bullet$	n	g_1	e_1	e_2
n	n	g_1	e_1	e_2
g_1	g_1	n	e_2	e_1
e_1	e_1	e_2	g_1	n
e_2	e_2	e_1	n	g_1

Mathematicians call D_2 a Klein group, which is identical to one of the two possible groups of 4 elements; the other is $\mathbb{Z}_4$, which has the following table:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

This table has some similarities to the passing of time being measured in quarters of an hour. We'll take a look at time and clocks in the next section.

Let's see now if there is any model for the Klein group. Consider this scenario: we have before us 2 coins, which can show heads or tails. There are three possible movements:

n: we don't touch anything;
g_1: we turn both coins over together;
e_1: we turn only the left-hand one;
e_2: we turn only the right-hand one

The table for this set of movements corresponds to the table of the Klein group.

The Klein group is also a symmetry group of a geometric figure, specifically that of any rectangle with unequal sides: the symmetries are:

1. Two rotations around the centre: one of $0 \pm 2\pi n$ radians (the identity) and another of $\pi \pm 2\pi n$ radians, $n \in \mathbb{Z}$.
2. Two axial reflections, of axes that pass through the centre and are parallel to each side.

If we make up the table, it turns out to be a group that is isomorphic to D_2. We'll look at the concept of isomorphism in more detail later on.

During the course of the transition from square to rectangle, some symmetry is lost – a square is more symmetrical than a rectangle. The group of the square is D_4, which has 8 elements, and the rectangle's group is D_2, which only has 4.

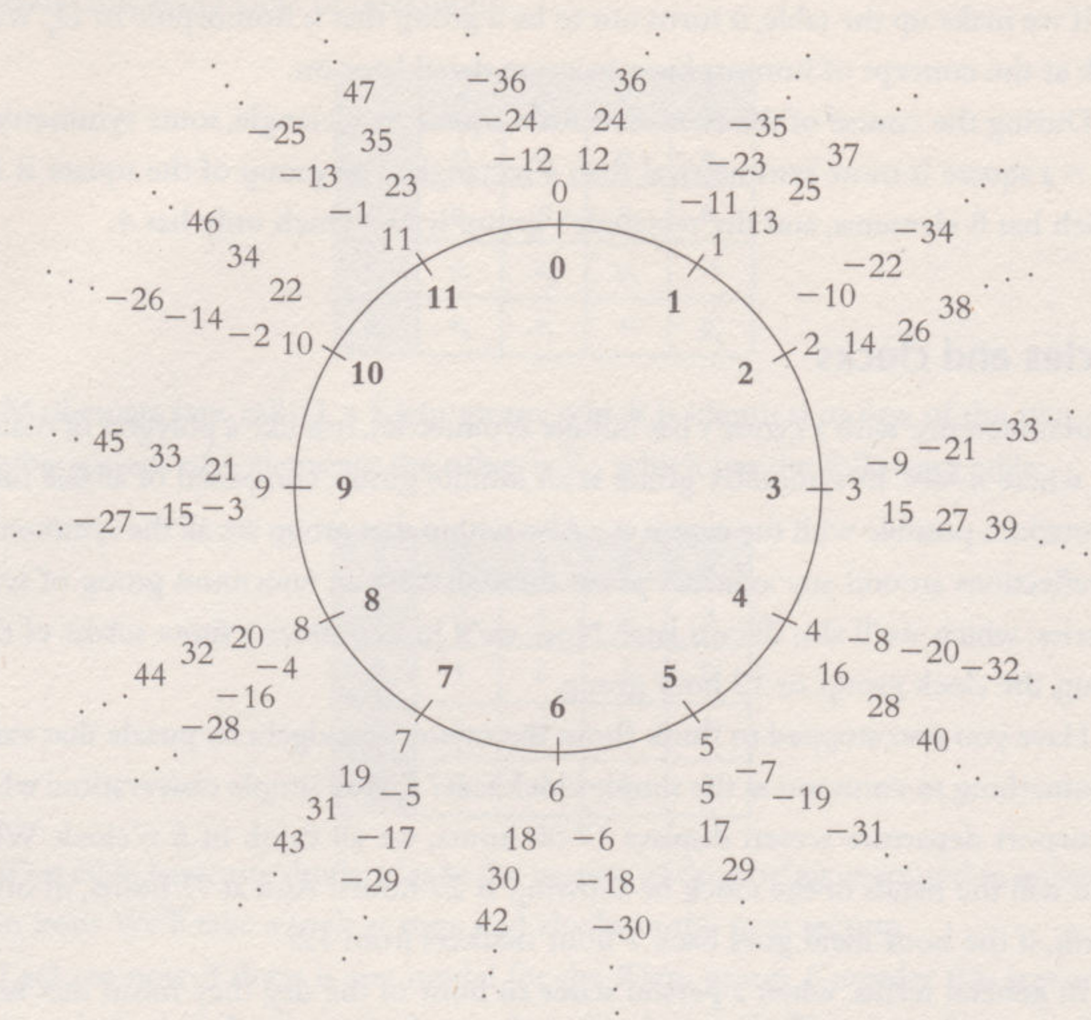
Cycles and clocks

A circumference with a centre c has infinite symmetries. It is like a polygon of n sides, but where $n \to \infty$. Its symmetry group is an infinite group, composed of all the turns or rotations possible with the centre at c. Also within that group are all the symmetries or reflections around any axis that passes through c. It's an enormous group of symmetries, which we'll also discuss later. Now we'll just consider a finite subset of that group, the clock group, or 12 hour group.

Have you ever stopped to think about the mysterious algebraic puzzle that exists in something so common as the simple clock face? A very simple observation: when an airport departure screen displays 17.00 hours, we all think of 5 o'clock. What time will the hands of the clock be showing at 29 hours? And at -7 hours, in other words, if the hour hand goes back 7 hour markers from 12?

In general terms, when a person states an hour of the day, they mean that hour and all those that differ from it at a multiple of 12. Let's move forward with a bit more precision. If $\mathbb{Z}$ designates the set of whole numbers, the following are the 12 subsets of $\mathbb{Z}$ relating to the hours of the day and night, shown here in bold:

$$
\begin{aligned}
\mathbf{0} &= \{\ldots, -24, -12, 0, 12, 24, 36, \ldots\} \\
\mathbf{1} &= \{\ldots, -35, -23, -11, 1, 13, 25, \ldots\} \\
\mathbf{2} &= \{\ldots, -34, -22, -10, 2, 14, 26, \ldots\} \\
\mathbf{3} &= \{\ldots, -33, -21, -9, 3, 15, 27, \ldots\} \\
\mathbf{4} &= \{\ldots, -32, -20, -8, 4, 16, 28, \ldots\} \\
\mathbf{5} &= \{\ldots, -31, -19, -7, 5, 17, 29, \ldots\} \\
\mathbf{6} &= \{\ldots, -30, -18, -6, 6, 18, 30, \ldots\} \\
\mathbf{7} &= \{\ldots, -29, -17, -5, 7, 19, 31, \ldots\} \\
\mathbf{8} &= \{\ldots, -28, -16, -4, 8, 20, 32, \ldots\} \\
\mathbf{9} &= \{\ldots, -27, -15, -3, 9, 21, 33, \ldots\} \\
\mathbf{10} &= \{\ldots, -26, -14, -2, 10, 22, 34, \ldots\} \\
\mathbf{11} &= \{\ldots, -25, -13, -1, 11, 23, 35, \ldots\}
\end{aligned}
$$



Clock face showing hours and related integers.

Note that in each subset of $\mathbb{Z}$ are the hours a and b which verify

$$a - b = 12n \text{ for some } n \in \mathbb{Z}.$$

In this way, each subset could have been described by any of the numbers that it contains. For example, **9 = –3 = 21 = –15 = 33 = …**

Let's recap a little before we move on. We have divided the set of all the whole numbers, $\mathbb{Z}$, into twelve parts or subset, which we have called **0**, **1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**, **10** and **11**. All of them are disjunctive parts, with no elements in common. Each whole number is in one of these twelve parts and one only. Let's look at a specific case. Where is 79? As $79 = 12 \cdot 6 + 7$, the hour 79 will be in the same part as hour 7, as they differ in $12 \cdot 6$ hours, a multiple of 12. Any two numbers will be in the same part if they differ by a multiple of 12.

We can perform additions of the twelve subsets as set out here:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

The table corresponds to a kind of addition, which we have designated **+** and which is defined by the expression $\boldsymbol{a + b = c}$, where $\boldsymbol{c}$ is the part where we find $c = a + b$.

So, for example, **8 + 11 = 7**, because 8 + 11 = 19 and 19 is in subset **7**, which is the same as **19**:

$$\mathbf{7} = \{\ldots, -29, -17, -5, 7, 19, 31, \ldots\}$$
$$\mathbf{19} = \{\ldots, -29, -17, -5, 7, 19, 31, \ldots\}$$

That is equivalent to counting 11 hours clockwise starting from 8, which means arriving at position 7. Every 12 hours we circle the clock face and start again.

We have built the table of a group, the group of the hours of a 12-hour clock. We could just the same have built a group of 24 hours. Let's call the group of hours $\mathbb{Z}_{12}$, like any good algebraist would. Any set $\mathbb{Z}_n$ would be built in a similar way, whatever the number $n > 0$ was. If the Lagrangian theorem is applied to find its subgroups, as $\mathbb{Z}_{12}$ has 12 elements, its possible subgroups will be of order 12, 6, 4, 3, 2 and 1. The subgroups of 12 and 1 elements are $\mathbb{Z}_{12}$ itself and form the trivial subgroup, so it is not necessary to list them. The tables are shown below with all subgroups of order 6, 4, 3 and 2 enclosed in boxes.

Order 6:

$\boxed{+}$	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11
$\boxed{0}$	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11
1	1	2	3	4	5	6	7	8	9	10	11	0
$\boxed{2}$	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{0}$	1
3	3	4	5	6	7	8	9	10	11	0	1	2
$\boxed{4}$	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{0}$	1	$\boxed{2}$	3
5	5	6	7	8	9	10	11	0	1	2	3	4
$\boxed{6}$	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5
7	7	8	9	10	11	0	1	2	3	4	5	6
$\boxed{8}$	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7
9	9	10	11	0	1	2	3	4	5	6	7	8
$\boxed{10}$	$\boxed{10}$	11	$\boxed{0}$	1	$\boxed{2}$	3	$\boxed{4}$	5	$\boxed{6}$	7	$\boxed{8}$	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Order 4:

+	$\boxed{0}$	1	2	$\boxed{3}$	4	5	$\boxed{6}$	7	8	$\boxed{9}$	10	11
$\boxed{0}$	$\boxed{0}$	1	2	$\boxed{3}$	4	5	$\boxed{6}$	7	8	$\boxed{9}$	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
$\boxed{3}$	$\boxed{3}$	4	5	$\boxed{6}$	7	8	$\boxed{9}$	10	11	$\boxed{0}$	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
$\boxed{6}$	$\boxed{6}$	7	8	$\boxed{9}$	10	11	$\boxed{0}$	1	2	$\boxed{3}$	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
$\boxed{9}$	$\boxed{9}$	10	11	$\boxed{0}$	1	2	$\boxed{3}$	4	5	$\boxed{6}$	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

Order 3:

+	$\boxed{0}$	1	2	3	$\boxed{4}$	5	6	7	$\boxed{8}$	9	10	11
$\boxed{0}$	$\boxed{0}$	1	2	3	$\boxed{4}$	5	6	7	$\boxed{8}$	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
$\boxed{4}$	$\boxed{4}$	5	6	7	$\boxed{8}$	9	10	11	$\boxed{0}$	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
$\boxed{8}$	$\boxed{8}$	9	10	11	$\boxed{0}$	1	2	3	$\boxed{4}$	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

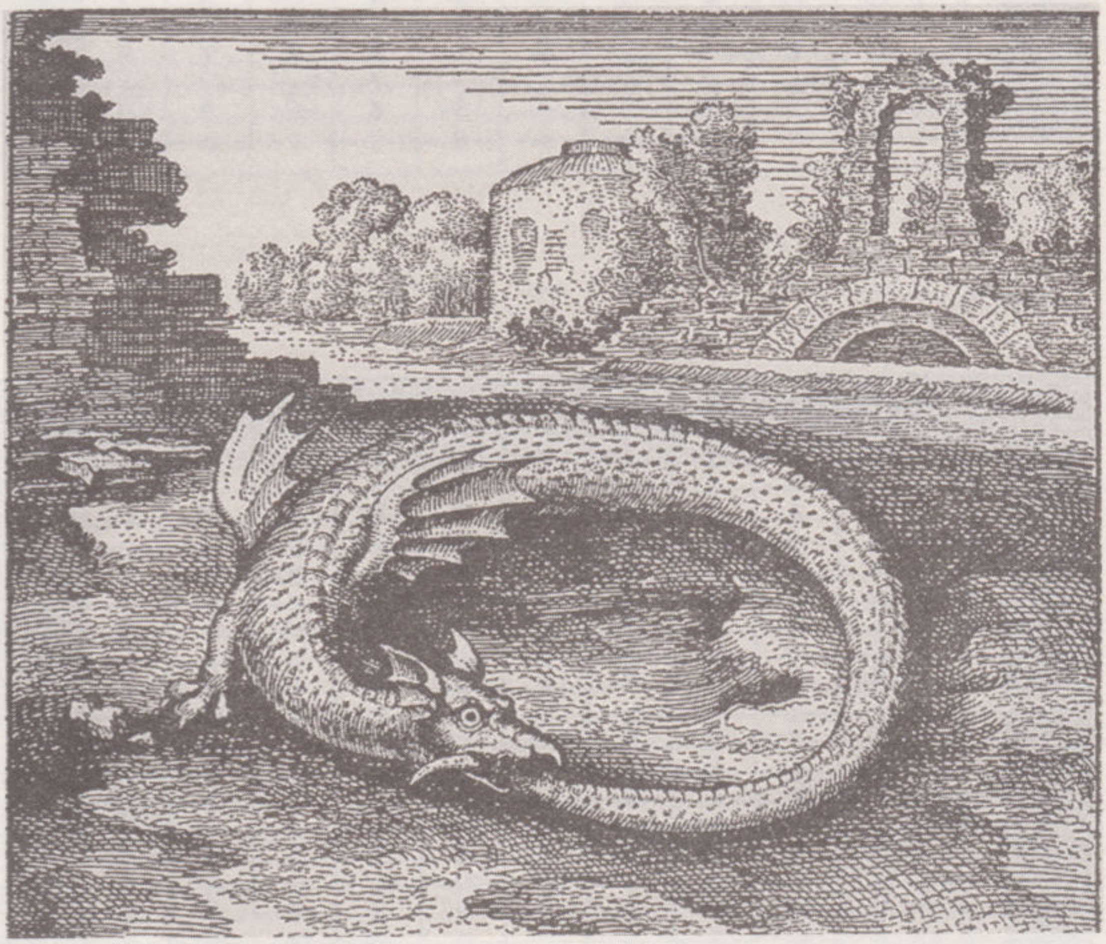
Order 2:

+	$\boxed{0}$	1	2	3	4	5	$\boxed{6}$	7	8	9	10	11
$\boxed{0}$	$\boxed{0}$	1	2	3	4	5	$\boxed{6}$	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
$\boxed{6}$	$\boxed{6}$	7	8	9	10	11	$\boxed{0}$	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

The second table corresponds to the group of quarter-hours. These are the subsets in which the large hands point to the quarter-hours at 3, 6 and 9. The subgroup with two elements is the subset of half-hours.

In reality, the element **1** 'generates' the group in the sense that by adding it again and again, you arrive at any other element. All groups of the $\mathbb{Z}_n$ kind derive from one element, which creates the others by tirelessly operating on itself. The group is like a column which, when it comes to the top, starts off again at the bottom; it has no beginning and no end, like the mythical serpent Ouroboros.

$\mathbb{Z}_n$ groups are therefore cyclic, and they are also known as C_n groups. The convention is to use the first notation ($\mathbb{Z}_n$) if the operation is designated as • or ·, and the second one (C_n) is kept for when the operation is designated as +. They are always groups of order n.



The serpent Ouroboros devoured itself in a way not dissimilar to a cyclic group. (Illustration from Atalanta Fugiens, *a book of alchemical emblems by Michael Maier, published in 1617.)*

Isomorphisms and other morphisms

Let's now take a better look at two of the previous groups. The first would be the rotations of the equilateral triangle.

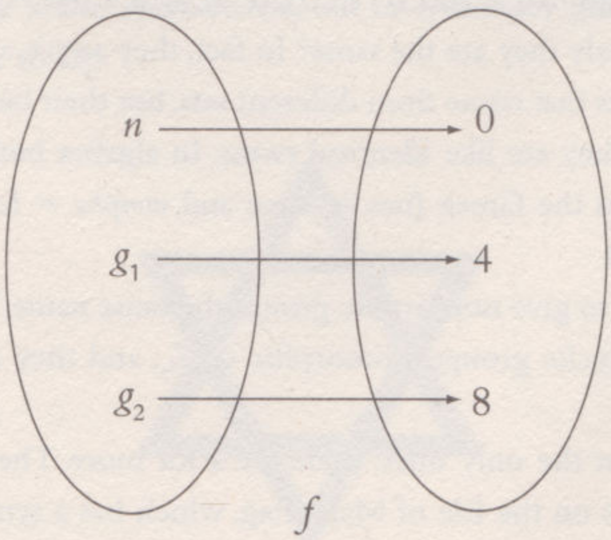
•	n	g_1	g_2
n	n	g_1	g_2
g_1	g_1	g_2	n
g_2	g_2	n	g_1

And the second, the group of order 3 from the $\mathbb{Z}_{12}$ subsets that we've just been talking about:

+	0	4	8
0	0	4	8
4	4	8	0
8	8	0	4

Do you see a certain similarity? The truth is that they seem the same, except that they are not written with the same symbols nor do they correspond to the same situations. One comes from the symmetries of a triangle, and the other from the face of a clock. If, between the first set, which we'll call *A*, and the second, which we'll call *B*, we establish a one-to-one correspondence, which we'll call *f*, in all cases it is verified that

$$f(a \bullet b) = f(a) + f(b).$$

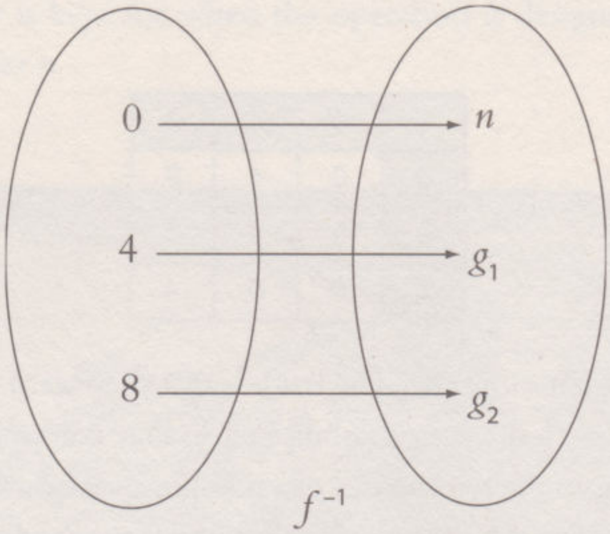


Correspondence f.

This is a somewhat complicated but nevertheless mathematically irrefutable way of saying that the operation • is preserved on going from left to right by changing into the operation +.

Or, in other words, it doesn't matter where the operation is verified: if we do it on the left with • and then through *f* we pass the result to the right, it's the same as passing each element through *f* to the right and to operate with + there. Since the correspondence is one-to-one, *f* has an inverse correspondence f^{-1} and also

$$f^{-1}(a + b) = f^{-1}(a) \bullet f^{-1}(b).$$



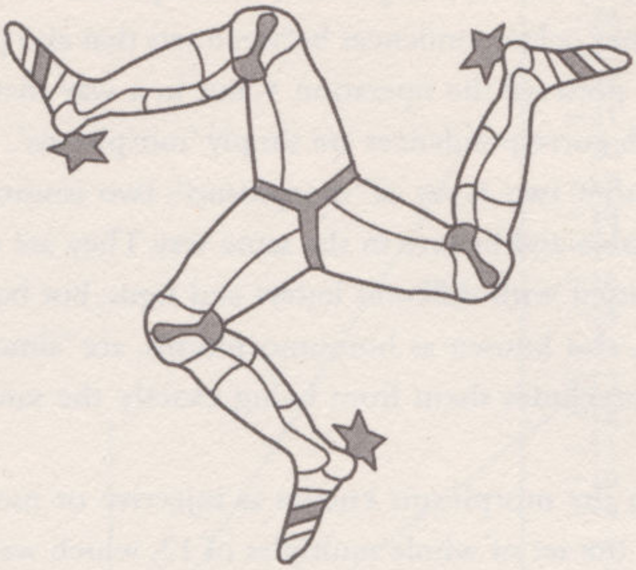
Inverse correspondence.

A and *B*, each one with its operation, • and +, behave in exactly the same way as groups.

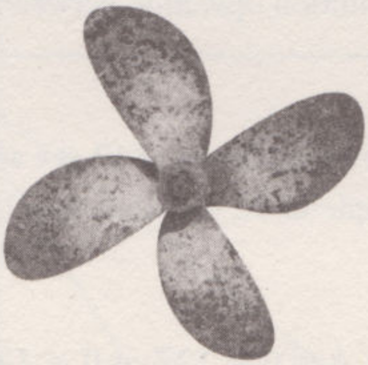
When two groups like *A* and *B* can relate in such a way – in short they behave the same – then surely they are the same? In fact, they aren't, as they are composed of different elements that come from different sets, but their behaviour makes them indistinguishable. They are like identical twins. In algebra both groups are said to be isomorphic, from the Greek (*isos* = same and *morphe* = form). It is frequently expressed as $A \cong B$.

It is the custom to give isomorphic groups the same name. The two groups that we have used are cyclic groups, isomorphic to $\mathbb{Z}_3$, and they are often designated with that name.

But they are not the only ones; there are a lot more. These include the symmetry group of legs on the Isle of Man's flag, which has a symmetry group that is also isomorphic to $\mathbb{Z}_3$:



The following image is also symmetrical and cyclic, though its purpose is nautical rather than mathematical. The propeller's symmetry group is isomorphic to $\mathbb{Z}_4$. It has only 4 elements:



This third image is also symmetrical, but its symmetry group has 12 elements, and it is isomorphic to D_6:

But, although they are very important, isomorphisms are not the end of the story. There are other correspondences between sets that also preserve the 'shape'; that is to say, they preserve the operation •, but in a way that is less robust than isomorphism. Such correspondences are simply 'morphisms'.

There are another two types of 'morphisms': two isomorphic groups have exactly the same table and behave in the same way. They are only different in so far as they are written with different letters and signs, but basically they are the same. Morphisms, also known as homomorphisms, are 'almost' isomorphic. An excess or deficit precludes them from being exactly the same. We'll show two examples below.

Let's start with the morphisms known as injective or monomorphisms. For example, consider the set of whole multiples of 12, which we'll call $12\mathbb{Z}$:

$$12\mathbb{Z} = \{\ldots, -36, -24, -12, 0, 12, 24, 36, \ldots\}.$$

In $12\mathbb{Z}$, the rules of the addition operation between the multiples of 12 follows these conditions:

1. If $a = 12n$ and $b = 12m$ are multiples of 12, the addition $a + b = 12n + 12m = 12\,(n + m)$ is also a multiple of 12 and therefore belongs to $12\mathbb{Z}$. In $12\mathbb{Z}$ the operation $+$ is closed.

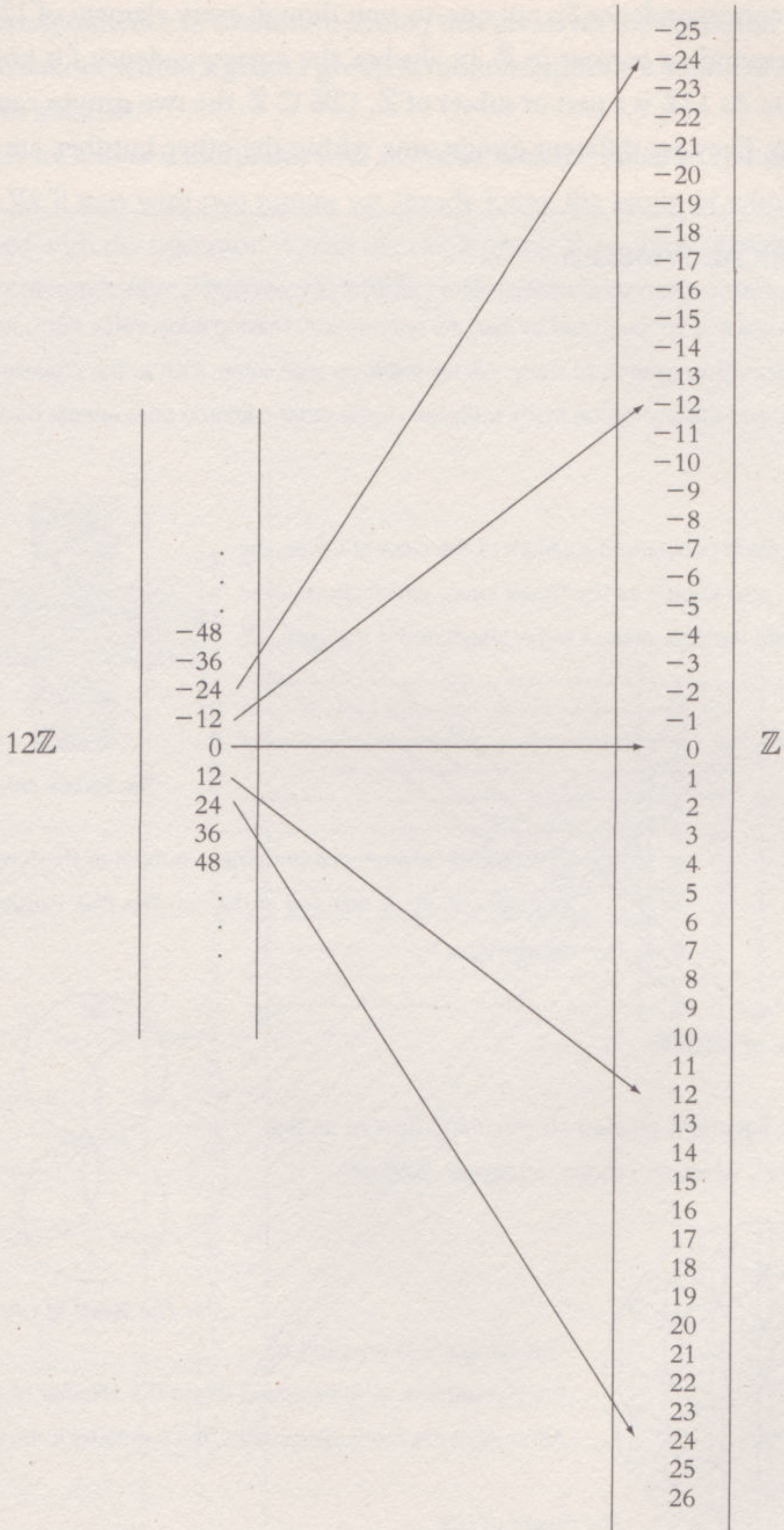
2. 0 is the neutral element. And $0 \in 12\mathbb{Z}$, as $0 = 12 \cdot 0$ and $0 \in \mathbb{Z}$.

3. The opposite of a multiple of 12 like $a = 12n$ is $-a = 12\,(-n)$, which is also a multiple of 12.

4. The associative law is trivially verified as, if it is valid for every $\mathbb{Z}$, there is even more reason for it to be valid for a subset of $\mathbb{Z}$ like $12\mathbb{Z}$.

It is perfectly possible to 'insert' $12\mathbb{Z}$ inside $\mathbb{Z}$ in a perfectly natural way:

$$f\,(a + b) = 12\,(a + b) = 12a + 12b = f(a) + f(b).$$



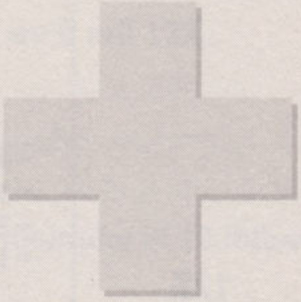
Inclusion of $12\mathbb{Z}$.

Now the correspondence f is not one-to-one, though every element of $12\mathbb{Z}$ does have its corresponding partner in $\mathbb{Z}$. In algebra, the correspondence f is known as monomorphic. As $12\mathbb{Z}$ is a part or subset of $\mathbb{Z}$, $12\mathbb{Z} \subset \mathbb{Z}$, the two groups cannot be isomorphisms. They are different groups, one within the other, but they are united

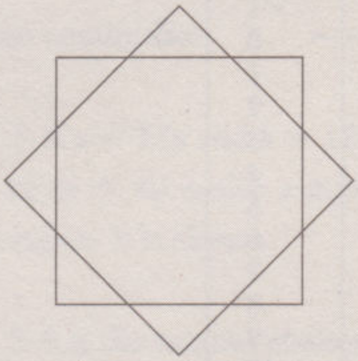
SYMMETRICAL SYMBOLS

Most of the signs and symbols used by man are symmetrical. Leaving aside traffic signs, we still have many identifying symbols to study. We have already seen some, such as the propeller, the Star of David, and the legs of the Manx triskelion. Some other common ones appear below.

The Da Vinci Code popularised a variant of the cross of Christ, the square cross, also known as the Greek cross, which shows symmetry D_4. Note that the plane can be tessellated if the arms are thick enough.



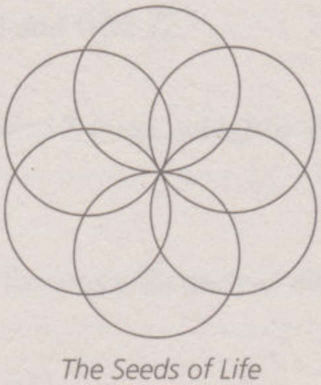
The square cross



The Star of Lakshmi

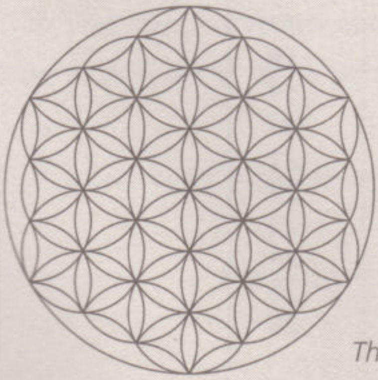
The Star of Lakshmi is a prominent symbol in Hinduism. In popular culture, it featured in the film *The Pink Panther*. Its symmetry is D_8.

The Ancient Egyptians created an ornament known as the "Seeds of Life", which also shows hexagonal symmetry.



The Seeds of Life

This symbol was adopted by the Phoenicians, who extended it into the "Flower of Life". Although it has more adornments, the symmetry is the same.

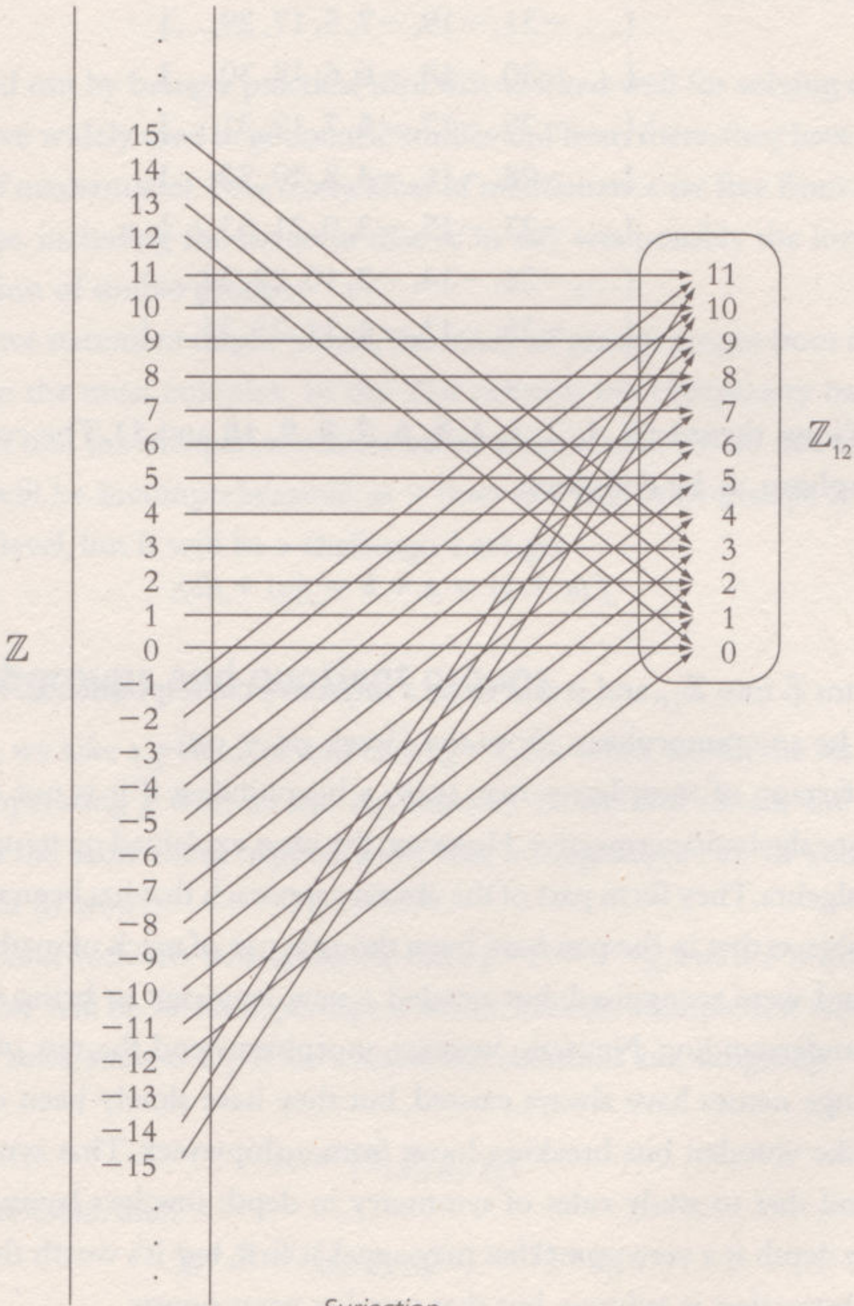


The Flower of Life

by a correspondence or monomorphism f that preserves the operation of $12\mathbb{Z}$ when it is carried out within a greater group. A monomorphism is a kind of isomorphism, but with a deficit.

Next we'll show a morphism with an excess, which is called an epimorphism in the jargon. We'll start with two groups we already know, the group of whole numbers $\mathbb{Z}$ equipped with the operation $+$, and the clock group $\mathbb{Z}_{12}$, equipped with its operation $+$. The correspondence between $\mathbb{Z}$ and $\mathbb{Z}_{12}$ will now be a correspondence f defined by

$$f(a) = a.$$



Surjection

The correspondence is clearly not one-to one; in fact, complete subsets of $\mathbb{Z}$ correspond to the same element of $\mathbb{Z}_{12}$. The observant reader will have noticed that the parts of $\mathbb{Z}$ that connect to an element of $\mathbb{Z}_{12}$ are our old friends from before:

$$\{\ldots, -36, -24, -12, 0, 12, 24, \ldots\}$$
$$\{\ldots, -35, -23, -11, 1, 13, 25, \ldots\}$$
$$\{\ldots, -34, -22, -10, 2, 14, 26, \ldots\}$$
$$\{\ldots, -33, -21, -9, 3, 15, 27, \ldots\}$$
$$\{\ldots, -32, -20, -8, 4, 16, 28, \ldots\}$$
$$\{\ldots, -31, -19, -7, 5, 17, 29, \ldots\}$$
$$\{\ldots, -30, -18, -6, 6, 18, 30, \ldots\}$$
$$\{\ldots, -29, -17, -5, 7, 19, 31, \ldots\}$$
$$\{\ldots, -28, -16, -4, 8, 20, 32, \ldots\}$$
$$\{\ldots, -27, -15, -3, 9, 21, 33, \ldots\}$$
$$\{\ldots, -26, -14, -2, 10, 22, 34, \ldots\}$$
$$\{\ldots, -25, -13, -1, 11, 23, 35, \ldots\}$$

And f joins them to **0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10** and **11**. The correspondence f is a morphism, as, by definition

$$f(a+b) = \boldsymbol{a} + \boldsymbol{b} = f(a) + f(b);$$

f transforms $\mathbb{Z}$ into $\mathbb{Z}_{12}$, and is said to be a surjective correspondence. The morphism is said to be an epimorphism (from the Greek *epi* = on).

This farrago of morphisms may seem a bit pointless if it is not seen from the appropriate algebraic perspective. However, the ideas explained up to now are basic in modern algebra. They form part of the abstract approach that has been used to analyse murky subjects that in the past have been thought out of reach of mathematics. They existed and were recognised, but needed a new language to bring them into the light of understanding. Neutrals, inverses, morphisms and the rest of the elements with strange names have always existed, but they have slowly been coming to the surface, like wooden bits breaking loose from a shipwreck. That symmetries form groups and that to study rules of symmetry in depth involves having to study the groups in depth is a viewpoint that may appal at first, but it's worth the trouble. The path of abstraction is arduous, but that way lies great power.

Chapter 2

What is a Group?

Like the ski resort full of girls hunting for husbands, and husbands hunting for girls, the situation is not as symmetrical as it might seem.

Alan Lindsay Mackay

Groups started out by being a practical tool that worked well for solving equations. Later, they were widely used in geometric studies and from there, they have ascended to the peak of mathematics. Few, if any, areas of mathematics are free from the influence of groups, including the theorem that in its day was possibly the longest ever: the classification of simple groups.

As the above statement might herald, the study of groups ranges from the easiest of concepts to the most complex. In the 21st century, the complexity has reached such extremes that the inexperienced are easily intimidated. We do not expect that this chapter will be incomprehensible, as it does not deal with groups at the most stratospheric level, but it will be a challenge. Let's go.

Normal subgroups and quotient groups

In a G group, we take a g element and form $g \bullet G$, in other words, the set of all the results from operating g with all the elements of G. We will obtain the G group itself, but will the same thing happen if we take a subgroup, F, of G? You can give it a go yourself (it won't).

If F is Abelian, both sets will definitely fulfil $g \bullet F = F \bullet g$, but if F is not Abelian, perhaps equality will be verified, perhaps it won't. Let's do a simple test with a group we've already seen, such as D_4. If we choose the rotations as a subgroup

$$R = \{n, g_1, g_2, g_3\}$$

and we use the table, then

$$g_i \bullet R = \{g_1, g_2, g_3, n\} = R \bullet g_i$$

for any rotation g_i (including $g_0 = n$) and

$$e_j \bullet R = \{e_1, e_2, e_3, e_4\} = R \bullet e_j$$

for any axial rotation e_j. The result is two distinct subsets. What is more, it was to be expected that the same set would be obtained operating from right to left, as R is a commutative subgroup. However, take note and believe us when we say that it is not always like that, and the products created to the right are not the same as the products created to the left. In general,

$$g \bullet F \neq F \bullet g.$$

An example extracted from a group of 120 elements would enable this to be seen. However, the confirmation process would be very laborious and it will save us a lot of work if we just believe it. Incidentally, the subsets $g \bullet F$ are known as lateral classes of F.

A subgroup, N, of a group, G, which fulfils the condition

$$g \bullet N = N \bullet g$$

for all the $g \in G$, is said to be a normal subgroup; the normal subgroups are also sometimes called "distinguished", but we only mention this in case you ever come across the term, which appears particularly in texts with a French origin.

Let's give another example, which comes from something we have already looked at: the clock group. There we had a large group, $\mathbb{Z}$, and a smaller subgroup, $12\mathbb{Z}$, formed by the integers that are multiples of 12

$$12\mathbb{Z} = \{\ldots, -36, -24, -12, 0, 12, 24, 36, \ldots\}.$$

This subgroup is Abelian, in other words, normal. The operation converting $12\mathbb{Z}$ into a group is the operation $+$. If $a = 12n$ and $b = 12m$ are multiples of 12, $a + b = 12n + 12m = 12(n + m) = c$; c is also a multiple of 12, and therefore, $c \in 12\mathbb{Z}$. We proved the rest of the group conditions previously. Instead of G, N and $\bullet$, we are talking about $\mathbb{Z}$, $12\mathbb{Z}$ and $+$. It's like a dictionary of equivalences.

Let's go on with these equivalences: what are the subsets of G that we have called $g \bullet N$ or lateral classes of N? Continuing with the example, and in accordance with the dictionary of equivalences, we are looking for the subsets of whole numbers, $\mathbb{Z}$, which we have denominated $g + 12\mathbb{Z}$. Let's build them:

$$
\begin{aligned}
&\dots\\
-11 + 12\mathbb{Z} &= \{\dots, -35, -23, -11, 1, 13, 25, \dots\}\\
-10 + 12\mathbb{Z} &= \{\dots, -34, -22, -10, 2, 14, 26, \dots\}\\
-9 + 12\mathbb{Z} &= \{\dots, -33, -21, -9, 3, 15, 27, \dots\}\\
-8 + 12\mathbb{Z} &= \{\dots, -32, -20, -8, 4, 16, 28, \dots\}\\
-7 + 12\mathbb{Z} &= \{\dots, -31, -19, -7, 5, 17, 29, \dots\}\\
-6 + 12\mathbb{Z} &= \{\dots, -30, -18, -6, 6, 18, 30, \dots\}\\
-5 + 12\mathbb{Z} &= \{\dots, -29, -17, -5, 7, 19, 31, \dots\}\\
-4 + 12\mathbb{Z} &= \{\dots, -28, -16, -4, 8, 20, 32, \dots\}\\
-3 + 12\mathbb{Z} &= \{\dots, -27, -15, -3, 9, 21, 33, \dots\}\\
-2 + 12\mathbb{Z} &= \{\dots, -26, -14, -2, 10, 22, 34, \dots\}\\
-1 + 12\mathbb{Z} &= \{\dots, -25, -13, -1, 11, 23, 35, \dots\}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
0 + 12\mathbb{Z} &= \{\dots, -36, -24, -12, 0, 12, 24, \dots\}\\
1 + 12\mathbb{Z} &= \{\dots, -35, -23, -11, 1, 13, 25, \dots\}\\
2 + 12\mathbb{Z} &= \{\dots, -34, -22, -10, 2, 14, 26, \dots\}\\
3 + 12\mathbb{Z} &= \{\dots, -33, -21, -9, 3, 15, 27, \dots\}\\
4 + 12\mathbb{Z} &= \{\dots, -32, -20, -8, 4, 16, 28, \dots\}\\
5 + 12\mathbb{Z} &= \{\dots, -31, -19, -7, 5, 17, 29, \dots\}\\
6 + 12\mathbb{Z} &= \{\dots, -30, -18, -6, 6, 18, 30, \dots\}\\
7 + 12\mathbb{Z} &= \{\dots, -29, -17, -5, 7, 19, 31, \dots\}\\
8 + 12\mathbb{Z} &= \{\dots, -28, -16, -4, 8, 20, 32, \dots\}\\
9 + 12\mathbb{Z} &= \{\dots, -27, -15, -3, 9, 21, 33, \dots\}\\
10 + 12\mathbb{Z} &= \{\dots, -26, -14, -2, 10, 22, 34, \dots\}\\
11 + 12\mathbb{Z} &= \{\dots, -25, -13, -1, 11, 23, 35, \dots\}
\end{aligned}
}
$$

$$
\begin{aligned}
12 + 12\mathbb{Z} &= \{\dots, -36, -24, -12, 0, 12, 24, \dots\}\\
13 + 12\mathbb{Z} &= \{\dots, -35, -23, -11, 1, 13, 25, \dots\}\\
14 + 12\mathbb{Z} &= \{\dots, -34, -22, -10, 2, 14, 26, \dots\}\\
15 + 12\mathbb{Z} &= \{\dots, -33, -21, -9, 3, 15, 27, \dots\}\\
16 + 12\mathbb{Z} &= \{\dots, -32, -20, -8, 4, 16, 28, \dots\}\\
17 + 12\mathbb{Z} &= \{\dots, -31, -19, -7, 5, 17, 29, \dots\}\\
18 + 12\mathbb{Z} &= \{\dots, -30, -18, -6, 6, 18, 30, \dots\}\\
19 + 12\mathbb{Z} &= \{\dots, -29, -17, -5, 7, 19, 31, \dots\}\\
20 + 12\mathbb{Z} &= \{\dots, -28, -16, -4, 8, 20, 32, \dots\}\\
21 + 12\mathbb{Z} &= \{\dots, -27, -15, -3, 9, 21, 33, \dots\}\\
22 + 12\mathbb{Z} &= \{\dots, -26, -14, -2, 10, 22, 34, \dots\}\\
&\dots
\end{aligned}
$$

We could go on writing for ever, but it can already be seen that there are only 12 subsets of $\mathbb{Z}$ or distinct lateral classes of $12\mathbb{Z}$.

Two lateral classes $a + 12\mathbb{Z}$ and $b + 12\mathbb{Z}$ coincide when $a - b \in 12\mathbb{Z}$. In general terms, this would be written (read in the dictionary of equivalences, where + is changed for • and $12\mathbb{Z}$ for N) in the following way: two elements a and b from the G group are in the same class if, and only if, $a \bullet b^{-1} \in N$.

The different lateral classes in our example could be:

$$\begin{aligned}
0 + 12\mathbb{Z} &= \{\dots, -36, -24, -12, 0, 12, 24, \dots\} \\
1 + 12\mathbb{Z} &= \{\dots, -35, -23, -11, 1, 13, 25, \dots\} \\
2 + 12\mathbb{Z} &= \{\dots, -34, -22, -10, 2, 14, 26, \dots\} \\
3 + 12\mathbb{Z} &= \{\dots, -33, -21, -9, 3, 15, 27, \dots\} \\
4 + 12\mathbb{Z} &= \{\dots, -32, -20, -8, 4, 16, 28, \dots\} \\
5 + 12\mathbb{Z} &= \{\dots, -31, -19, -7, 5, 17, 29, \dots\} \\
6 + 12\mathbb{Z} &= \{\dots, -30, -18, -6, 6, 18, 30, \dots\} \\
7 + 12\mathbb{Z} &= \{\dots, -29, -17, -5, 7, 19, 31, \dots\} \\
8 + 12\mathbb{Z} &= \{\dots, -28, -16, -4, 8, 20, 32, \dots\} \\
9 + 12\mathbb{Z} &= \{\dots, -27, -15, -3, 9, 21, 33, \dots\} \\
10 + 12\mathbb{Z} &= \{\dots, -26, -14, -2, 10, 22, 34, \dots\} \\
11 + 12\mathbb{Z} &= \{\dots, -25, -13, -1, 11, 23, 35, \dots\}
\end{aligned}$$

Now that we have defined the normal subgroups, we can move on to the next concept, the quotient group. If we start off from a G group and from a normal subgroup N, for any $g \in G$ we can consider the subsets or parts of G, called lateral classes of N, defined by $g \bullet N$, which incidentally fulfil $g \bullet N = N \bullet g$. Let's give the name G/N to this new set.

$$G/N = \{g \bullet N, g \in G\}.$$

To simplify it, let's give the denomination $\mathbf{g}$ to the lateral classes

$$\mathbf{g} = g \bullet N.$$

The next obvious step is to provide this new set with a • operation and with a group structure. All that is required is to put

$$\mathbf{g}_1 \bullet \mathbf{g}_2 = (g_1 \bullet g_2) \bullet N.$$

The example of clocks will make this question easier to understand. Let's study the quotient group $\mathbb{Z}/12\mathbb{Z}$; the lateral classes of $12\mathbb{Z}$ are:

$$\begin{aligned}
0 + 12\mathbb{Z} &= \{\ldots, -36, -24, -12, 0, 12, 24, \ldots\} = \mathbf{0}\\
1 + 12\mathbb{Z} &= \{\ldots, -35, -23, -11, 1, 13, 25, \ldots\} = \mathbf{1}\\
2 + 12\mathbb{Z} &= \{\ldots, -34, -22, -10, 2, 14, 26, \ldots\} = \mathbf{2}\\
3 + 12\mathbb{Z} &= \{\ldots, -33, -21, -9, 3, 15, 27, \ldots\} = \mathbf{3}\\
4 + 12\mathbb{Z} &= \{\ldots, -32, -20, -8, 4, 16, 28, \ldots\} = \mathbf{4}\\
5 + 12\mathbb{Z} &= \{\ldots, -31, -19, -7, 5, 17, 29, \ldots\} = \mathbf{5}\\
6 + 12\mathbb{Z} &= \{\ldots, -30, -18, -6, 6, 18, 30, \ldots\} = \mathbf{6}\\
7 + 12\mathbb{Z} &= \{\ldots, -29, -17, -5, 7, 19, 31, \ldots\} = \mathbf{7}\\
8 + 12\mathbb{Z} &= \{\ldots, -28, -16, -4, 8, 20, 32, \ldots\} = \mathbf{8}\\
9 + 12\mathbb{Z} &= \{\ldots, -27, -15, -3, 9, 21, 33, \ldots\} = \mathbf{9}\\
10 + 12\mathbb{Z} &= \{\ldots, -26, -14, -2, 10, 22, 34, \ldots\} = \mathbf{10}\\
11 + 12\mathbb{Z} &= \{\ldots, -25, -13, -1, 11, 23, 35, \ldots\} = \mathbf{11}
\end{aligned}$$

and they can be designated as they are written on the right in bold. If we define the sum of classes just as indicated above,

$$\boldsymbol{a} + \boldsymbol{b} = (a + b) + 12\mathbb{Z}$$

and the group has as this table:

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	[8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

it is none other (or it is isomorphic) than our old friend $\mathbb{Z}_{12}$. We can write, then

$$\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_{12}.$$

If we consider, for example, the square group, D_4, a normal subgroup is R, the subgroup of the rotations, the lateral classes are two, as:

$$\begin{aligned}
n \bullet R &= \{g_1, g_2, g_3, n\} = R \bullet n \\
g_1 \bullet R &= \{g_1, g_2, g_3, n\} = R \bullet g_1 \\
g_2 \bullet R &= \{g_1, g_2, g_3, n\} = R \bullet g_2 \\
g_3 \bullet R &= \{g_1, g_2, g_3, n\} = R \bullet g_3 \\
e_1 \bullet R &= \{e_1, e_2, e_3, e_4\} = R \bullet e_1 \\
e_2 \bullet R &= \{e_1, e_2, e_3, e_4\} = R \bullet e_2 \\
e_3 \bullet R &= \{e_1, e_2, e_3, e_4\} = R \bullet e_3 \\
e_4 \bullet R &= \{e_1, e_2, e_3, e_4\} = R \bullet e_4
\end{aligned}$$

and two classes are equal when $a \bullet b^{-1} \in R$.

If we give the denomination $\boldsymbol{n}$ and $\boldsymbol{e}$ to the two lateral classes

$$\begin{aligned}
\boldsymbol{n} &= \{g_1, g_2, g_3, n\} \\
\boldsymbol{e} &= \{e_1, e_2, e_3, e_4\}
\end{aligned}$$

applying the quotient group structure to those classes, the result is the table

•	*n*	*e*
n	*n*	*e*
e	*e*	*n*

that turns out to be a group of two elements, already known and isomorphic to $\mathbb{Z}_2$:

+	0	1
0	0	1
1	1	0

$$D_4/R \cong \mathbb{Z}_2.$$

What's more, this example can help to understand the reason behind the curious name "quotient group". It turns out that, among finite groups

$$|G/N| \cdot |N| = |G|$$

or, equally,

$$|G/N| = |G|/|N|.$$

The order of the quotient group is the quotient of the orders of the groups.

If a G group admits a chain of inclusions of subgroups, of the trivial subgroup up to G

$$\{n\} = G_1 \subset G_2 \subset G_3 \subset \ldots \subset G_n = G,$$

where

1. all the G_i (except the last one, G) are normal and
2. the successive quotient groups G_{i+1}/G_i are Abelian,

we say that G is soluble (as in 'solvable' ignore any chemical connotations).

We have come to the end of the road, the definition of a soluble group, and it has not been simple; so just imagine how difficult it must have been for algebraists who were contemporaries of Évariste Galois, the genius who started off this topic less than 200 years ago. And the most curious part of it all is that it all poured out of Galois' mind when he was working not with symmetries, but with equations.

The symmetry group

When it comes to finite groups, the one we could call 'father of all the groups' is the symmetry group: a group that is defined successively for each number n, which is written S_n and which has $n!$ elements, in other words, a monstrous number, as n grows.

For the moment, let's start by building the simplest ones. In general terms, S_n is the set of permutations of n objects. We'll make these objects the letters of the alphabet, although it would make no difference if we permutated letters, numbers, coloured balls or whatever. All the permutation groups are isomorphic. To start, let's write two permutations p_1 and p_2 of four objects.

$$p_1 = \begin{bmatrix} a & b & c & d \\ c & a & b & d \end{bmatrix}, p_2 = \begin{bmatrix} a & b & c & d \\ b & d & c & a \end{bmatrix}.$$

The operation of the group will be best defined by dealing with more permutations. Where will a end up? According to what is shown in p_2, a is going to end in b. And where will this b end up? Well, p_1 processes (or we could say *permutes*) it into a. That is to say, a finally ends up as a via b. Following through with the process to the end with the other objects we get a third permutation

$$p_3 = \begin{bmatrix} a & b & c & d \\ a & d & b & c \end{bmatrix}.$$

And we can then say $p_1 \bullet p_2 = p_3$. Operations between permutations consist of making one change (on the right first, remember) and then the other. The result is another permutation, which may be one of the two previous ones, or it may not.

The group structure becomes clear since the permutations are associative (by definition of being a physical permutation) and there is a privileged permutation:

$$n = \begin{bmatrix} a & b & c & d \\ a & b & c & d \end{bmatrix},$$

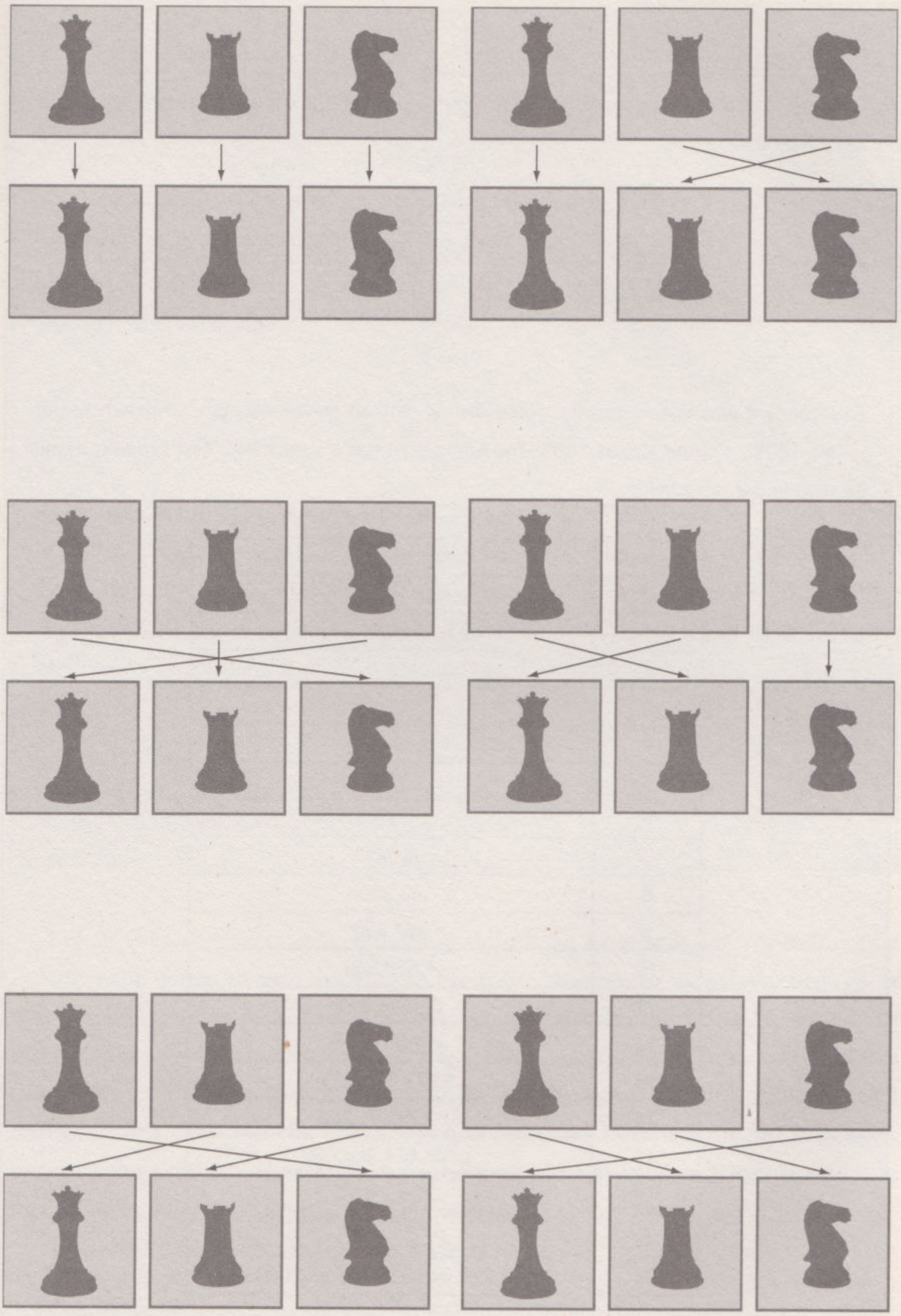
which consists of leaving everything just as it was, and which acts as a neutral element. There is also an inverse element. If

$$p = \begin{bmatrix} a & b & c & d \\ b & d & a & c \end{bmatrix}$$

the permutation can be inverted, which puts everything into reverse order (or back into its previous position):

$$p^{-1} = \begin{bmatrix} a & b & c & d \\ c & a & d & b \end{bmatrix},$$

and then $p \bullet p^{-1} = p^{-1} \bullet p = n$.



The permutation group for three objects.

$n!$

This curious term conceals a well-known arithmetical notation. If n is a positive integer, $n!$ signifies the product

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

and is known as n factorial or factorial of n. For calculation purposes that are unimportant here, the conventional value is used.

$$0! = 1.$$

This concept was introduced by a little-known French mathematician, Christian Kramp (1760–1826), as long ago as 1808. The number $n!$ has a surprisingly fast growth, as can be seen in the table below:

n	$n!$
0	1
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880
10	3,628,800
11	39,916,800
12	479,001,600
13	6,227,020,800
14	87,178,291,200
15	1,307,674,368,000
… 20 …	2,432,902,008,176,640,000
25	15,511,210,043,330,985,984,000,000

In fact, the formula known as Stirling's approximation gives a very good idea of the size of $n!$, while itself being a thing of beauty:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

The definition of $n!$ can be extended to any value of n, even if it isn't a positive integer, but that requires the tools of higher analysis and definition of the function Γ. But one of the best known uses of $n!$ is in the series

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$$

that enables the number e to be defined elegantly.

An amusing problem, solved by Jacob Bernoulli (1654–1705), goes like this: On arriving at a party, n guests hand their hat over at the entrance hall. What are the probabilities that when they leave nobody gets the right hat? Bernoulli worked out that the solution to "the hat problem" was

$$p_n = 1-\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}.$$

And all with no knowledge of the concept of factorial!

Let us return to S_n, as the following definition functions for any symmetry group: each permutation p of S_n has a sign, positive or negative, though it may not be noticeable at first sight. A transposition is a permutation that only interchanges two elements, one for the other. It is a simple fact, although not simple to demonstrate, that all permutations are the result of carrying out several transpositions, one after the other. When the number of transpositions is an even number, the sign of p is positive; when the number of transpositions is odd, the sign for p is negative.

The number of permutations of n elements is, as they teach in school, $n!$

$$|S_n| = n!$$

The set formed by all the positive transpositions is denoted by A_n and is called alternating group of order n. Naturally, $A_n \subset S_n$.

The alternating group is not Abelian when n > 3. As the number of positive transpositions is equal to that of the negative ones, $|A_n| = |S_n|/2$,

S_1 will be formed by all the possible permutations of a single letter, for example, a. This is not a problem, as there is only one permutation that transforms the expression "a" into "a", i.e. into itself. The group S_1 has just one element, and it is therefore the trivial group that has the neutral element only.

S_2 is the set of permutations of two elements, which consists of only two permutations:

$$n=\begin{bmatrix} a & b \\ a & b \end{bmatrix}, p=\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

and it is a cyclic group, isomorphic to $\mathbb{Z}_2$. Naturally, it is an Abelian group.

S_3 is isomorphic to the group of symmetries of an equilateral triangle. It has six elements, which interchange three letters, a, b, c, between them. $S_3 \cong D_3$. S_3 is not Abelian, therefore, and neither are the successive symmetry groups S_n.

S_4 coincides with the tetrahedron group of symmetries, the regular polyhedron of 4 vertices. It has 24 elements distributed in 12 rotations and 12 symmetries with torsion. S_4 is isomorphic to the proper rotations of a cube, and so a complete group of symmetries of a polyhedron is the subgroup of the symmetries of the next one.

S_5 is a different kettle of fish. Apart from already consisting of 120 permutations - quite a decent amount - S_5 is, as we shall see, the first non-soluble symmetry group, and all the ones that follow are not soluble either, hence Galois' famous result: there is no simple arithmetic formula that solves all the equations equal to or above the fifth degree.

Never fear, we will come back to look at S_5 and the fact that it is not a soluble group later on. But let's remember that this means there is no chain of normal subgroups that ends in S_5:

$$\{n\} = G_1 \subset G_2 \subset G_3 \subset \ldots \subset G_n = S_5$$

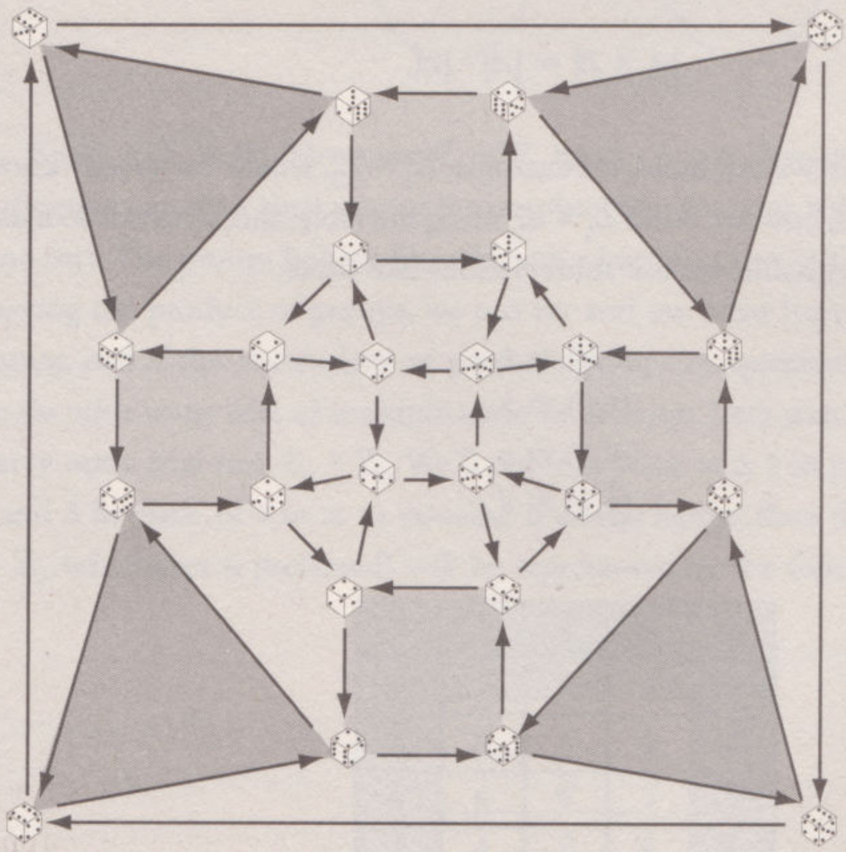
and whose intermediate quotient groups are Abelian. That means we have to look into the normal subgroups of S_5 as a first step to see if they are soluble or not.

Arthur Cayley (see panel, right) is credited with another important result concerning the symmetry group, which states that every finite group is isomorphic to some subgroup of a symmetry group.

ARTHUR CAYLEY (1821–1895)

One of the most significant mathematicians of all time spent 13 years of his life practising law to earn his living. Arthur Cayley was born in England but spent his early years in Russia, where his father was working. At Cambridge University, he would first show his worth. He graduated with a first in mathematics and showed a singular talent for languages. As well as English, Cayley spoke fluent Italian, German, French and Greek.

On finishing his studies, Cayley became a successful lawyer. As the years went by, there was an increase the financing of the Lucasian professorship at Cambridge, a position once held by Newton, and with the modest university salaries improved, Cayley was prompted to give up his career as a lawyer and devote his time to his true, but far less lucrative, vocation. His field was analysis, quaternions, matrices, determinants and *n*-dimensional geometry, though today he is revered above all for his work on group theory. It was Cayley who studied groups in abstract form and would bring them out of their lair in the world of equations to give them an independent existence and define them in a universal way. Together with his colleague and friend James Joseph Sylvester (1814–1897), Cayley formed a high-performance science team.



A Cayley graph showing the group of rotations of a die. The use of shading and generating elements helps to visualise the sometimes intricate structure of a group and makes it easier to handle. The group turns out to be isomorphic to S_4, which is not commutative. This indicates that, contrary to what happens on the plane, spatial rotations do not necessarily commute if they do not share the same axis.

Groups × groups = more groups

Groups behave like pieces of Lego. Starting with two or more pieces, more pieces can be made, and finally an object of varying complexity. A standard procedure makes it possible to pass from two groups to a larger compound group as follows: if A and B are groups with respective operations $\bullet$ and $\circ$, the product known as the Cartesian group product is $A \times B$,

$$A \times B = \{(a,b), \text{ with } a \in A, b \in B\},$$

formed by pairs of members of A and B, and this can be converted into a group in a simple way by defining a new operation $\star$ between pairs starting from the previous operations:

$$(a_1,b_1) \star (a_2,b_2) = (a_1 \bullet a_2, b_1 \circ b_2).$$

That is to say that, with the help of $\star$, a larger group can be defined from $A \times B$. Take care with the notations, as sometimes no link whatsoever is preserved between the resulting group and the original groups. On the other hand, it is verified that for finite groups

$$|A \times B| = |A| \cdot |B|.$$

With this procedure we can build, for example, $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we already know will have 4 elements. If, now, we build $\mathbb{Z}_2 \times \mathbb{Z}_2$ using the table, and, so as not to make it too big and unrecognisable, we use abbreviations like these:

$$(0,0) = a$$
$$(1,0) = b$$
$$(0,1) = c$$
$$(1,1) = d$$

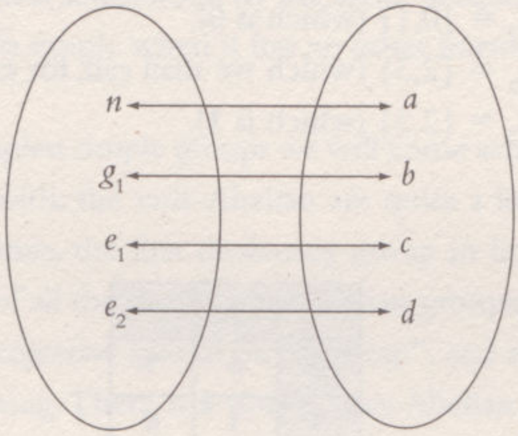
the result is the table:

•	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

If anyone is wondering what group this is, welcome back our old friend D_2 or the Klein group;

•	n	g_1	e_1	e_2
n	n	g_1	e_1	e_2
g_1	g_1	n	e_2	e_1
e_1	e_1	e_2	g_1	n
e_2	e_2	e_1	n	g_1

and establish the following isomorphism:



So $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_2$. However, $\mathbb{Z}_2 \times \mathbb{Z}_3$ has $2 \cdot 3 = 6$ elements, like D_3, but – what a disappointment – they are not isomorphic! The first one is Abelian and the second one isn't. The groups behave like children: charming, but, at times, annoying. Before leaving the product of groups, we can try and see what happens with the quotient group, and if the vocabulary of product-groups vs. quotient-groups has anything to do with some sort of multiplication or division. Let's start with an easy case and carry out a trial run: $\mathbb{Z}_2 \times \mathbb{Z}_2$. We'll put 0 in place of a, 1 in place of b, 2 in place of c and 3 in place of d so as to visualise it all the better; then the Klein group (or $\mathbb{Z}_2 \times \mathbb{Z}_2$, whichever is preferred) will be represented by the following table:

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

which is the "numerator" or product group, and $\mathbb{Z}_2$, we can see, is effectively one of its normal subgroups:

+	0	1
0	0	1
1	1	0

The lateral classes are:

$0 + \mathbb{Z}_2 = \{0,1\}$ (which we shall call, for example, **0**),
$1 + \mathbb{Z}_2 = \{0,1\}$ (which is **0**),
$2 + \mathbb{Z}_2 = \{2,3\}$ (which we shall call, for example, **1**),
$3 + \mathbb{Z}_2 = \{2,3\}$ (which is **1**),

with the table

+	**0**	**1**
0	**0**	**1**
1	**1**	**0**

that is evidently isomorphic to $\mathbb{Z}_2$, and so

$$\mathbb{Z}_2 \times \mathbb{Z}_2 / \mathbb{Z}_2 \cong \mathbb{Z}_2$$

and it seems that the group-numbers parallel works. And, in reality, it is not too complicated – though it is, certainly, somewhat laborious – to see that in general,

$$A \times B / A \cong B$$

Groups, groups, groups…

By now the sense is growing that there is a large number of groups. There are groups that are finite, and groups, like that of the symmetries of a sphere, that are infinite. There are groups that may be assumed to have, and do have, a fixed centre, such as that of a regular polygon or that of a cube, which are called point groups; and there are those with several, and even infinite, centres, which can be seen in decorative

friezes. There are groups that are isomorphic to products of other, simpler, groups and groups that aren't; and there are groups that have subgroups and groups that haven't, which are described as simple groups.

They are the true building blocks of the groups, those that enable all the others to be built. In the 21st century, we are now able to identify all of them, but the truth is that this statement involves a leap of faith, because the full demonstration of the structure of the simple groups is also called – and justifiably so – "the monster theorem". In its first draft, it took up 10,000 pages (some say 15,000; and others say a mere 5,000). Subsequent simplifications – up to now there have been three rounds – have brought it down to 6,000 pages, but we're still talking about a big book. So let's define what is understood by the term "simple group:" a group with more than one element is simple when it has no other normal subgroup than itself and its identity.

If we look for the simplest simple groups we will come across the Abelian groups, which are easy to deal with; the non-Abelian are quite a lot more difficult, with A_5, our recent acquaintance, the first dissenting group in first place. It is of order 60 and it is the smallest of all the simple non-Abelian groups; the next one answers to the fancy name of "projective special linear group", and is of order 168. But, as we shall see, that is nothing. There is a simple non-Abelian group (named "The Monster") of order 808,017,424,794,512,875,886,459,904,961,710,757,005,754, 368,000,000,000.

Chapter 3

Symmetry in More Dimensions

Picture yourself in a boat on a river, with tangerine trees and marmalade skies
Somebody calls you, you answer quite slowly, a girl with kaleidoscope eyes.
Lucy in the Sky with Diamonds The Beatles

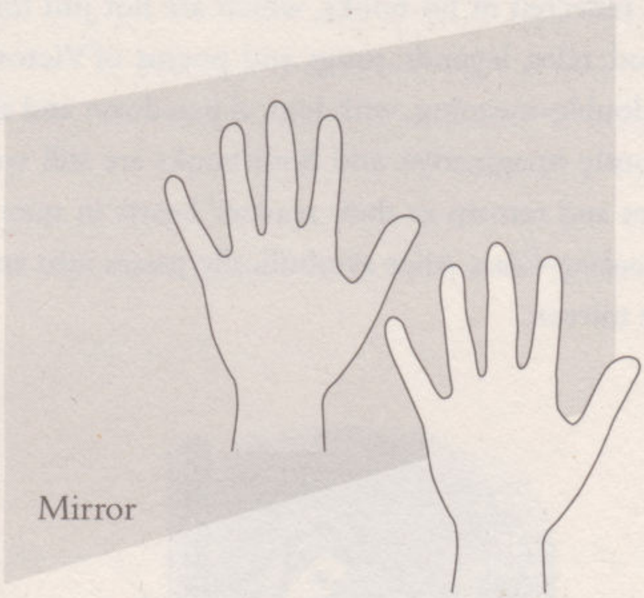
Through the Looking Glass is a literary work generally regarded as being the second part of *Alice's Adventures in Wonderland*. Its author, Lewis Carroll, a Church of England clergyman and photography enthusiast, was also a mathematician specialising in logic. This fact is reflected in his books, which are not just limited to depicting the typical childhood, tales, legends, songs and poems of Victorian England. The text is loaded with double-meaning, with logical paradoxes and situations as absurd as they are marvellously imaginative, and both books are still well stocked on the shelves of bookshops and remain in their readers' hearts in spite of the time gone by. In *Through the Looking-Glass,* Alice symbolically passes into another reality with the help of a simple mirror.

John Tenniel's illustration for Lewis Carroll's Through the Looking-Glass

Mirror, mirror [on the wall]

Mirrors figure in the origins of spatial symmetry with specular symmetry, which transforms every point into its mirror image.

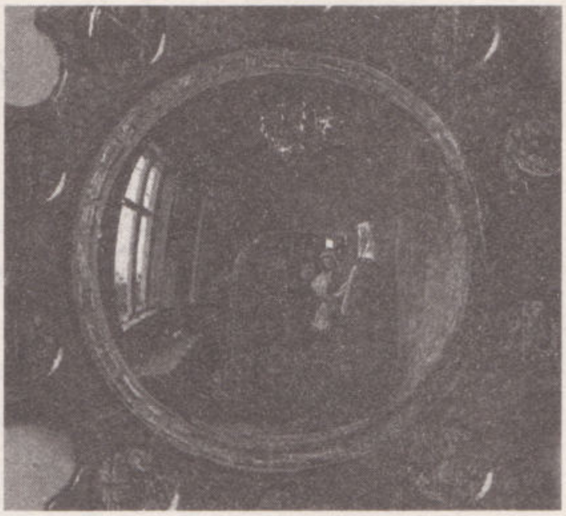
Specular symmetry involves a concept that is more or less new to our discussions so far: chirality, which geometricians sometimes call torsion. Its clearest example is a simple pair of gloves. One is the specular image of the other, and both gloves, the right and the left, are not congruent. It is not possible to put a right-hand glove on the left hand (and the same thing is true with the other hand), as the two gloves – or the hands – are chiral objects which do not coincide with their specular images. Although they have the same form, they do not share the same orientation. In one it is laevorotatory (oriented towards the left) and in the other it is dextrorotatory (oriented to the right.) It could be said that each glove is the 'anti-glove' of the other.



Specular symmetry showing chirality.

Proteins are chiral molecules, and the human body, for instance, consists only of laevorotatory proteins (and laevorotatory fats and carbohydrates for that matter). Why that should be is not a question within the field of group theory, so we'll leave it for the biologists to figure out.

Let's imagine we're in an everyday sort of situation: when we go to the hairdresser's, we usually get our hair cut in front of a wall fitted with mirrors. On concluding the task, and as a final touch, it is the custom for the hairdresser to seek the client's

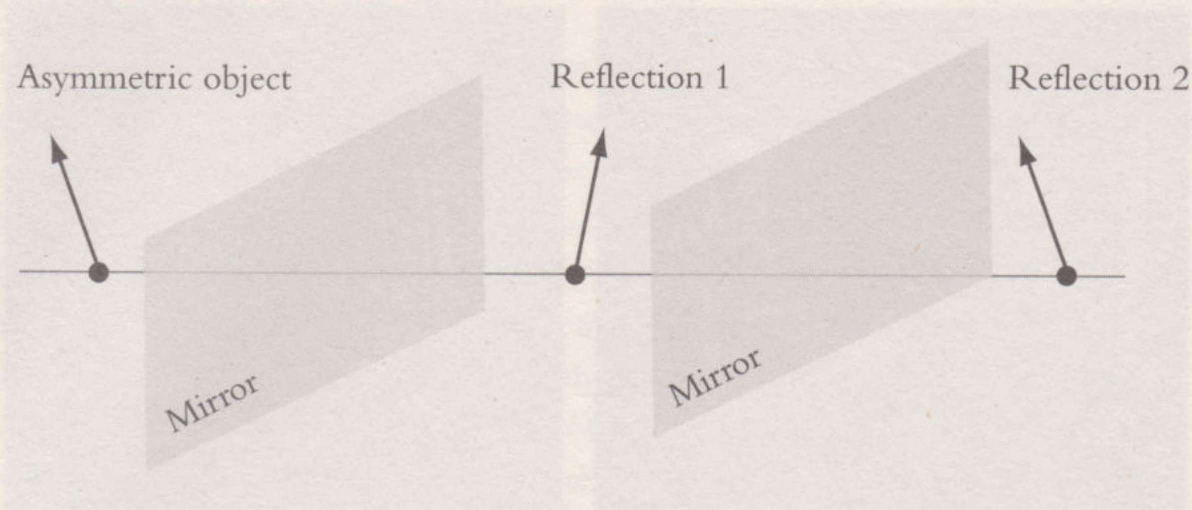


Jan Van Eyck's famous painting The Arnolfini Portrait, *from 1434. In the background, and measuring little more than 5 cm, a small mirror reflects all the room, a glimpse through the window, the backs of the famous couple and another two people located in the same place as the observer of the picture.*

approval of the haircut and for the client to see how the haircut looks from the back. The hairdresser holds a hand mirror and places it behind the client's head. The client looks at the image in the mirror which is in front of him or her (as he or she hasn't got eyes in the back of their head) and sees everything, including the image of the back of his or her head reflected in the second mirror.

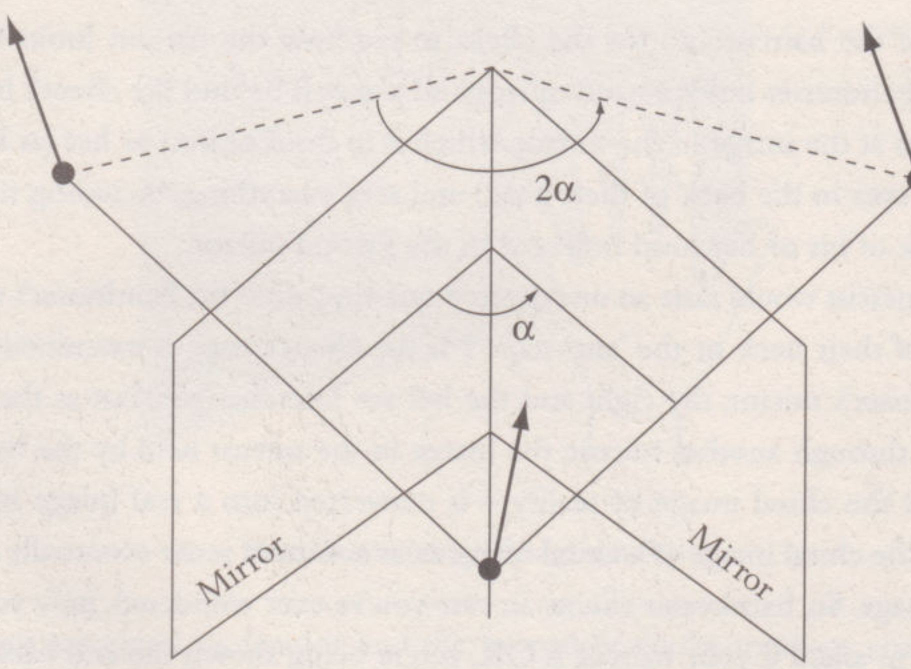
A geometrist would raise an unexpected question: does the hairdresser's client see the nape of their neck or the 'anti-nape'? If the client's nape is examined through the hairdresser's mirror, the right and the left are interchanged; but as the scene is examined through another mirror, the image in the mirror held by the hairdresser – which is the chiral image of reality – is converted into a real image in the big mirror, as the chiral image of a chiral image is, as common sense eventually tells us, a normal image. So, hairdresser clients, in case you've ever wondered, now you know: when you're asked if your haircut is OK, you're being shown the real back of your head, the one on view by the rest of us.

And what is the mathematical consequence? Well, that two specular symmetries, which present a torsion, are transformed, one after the other, into a simple symmetry without torsion, into a translation the range of which is the sum of the distances to each axis.

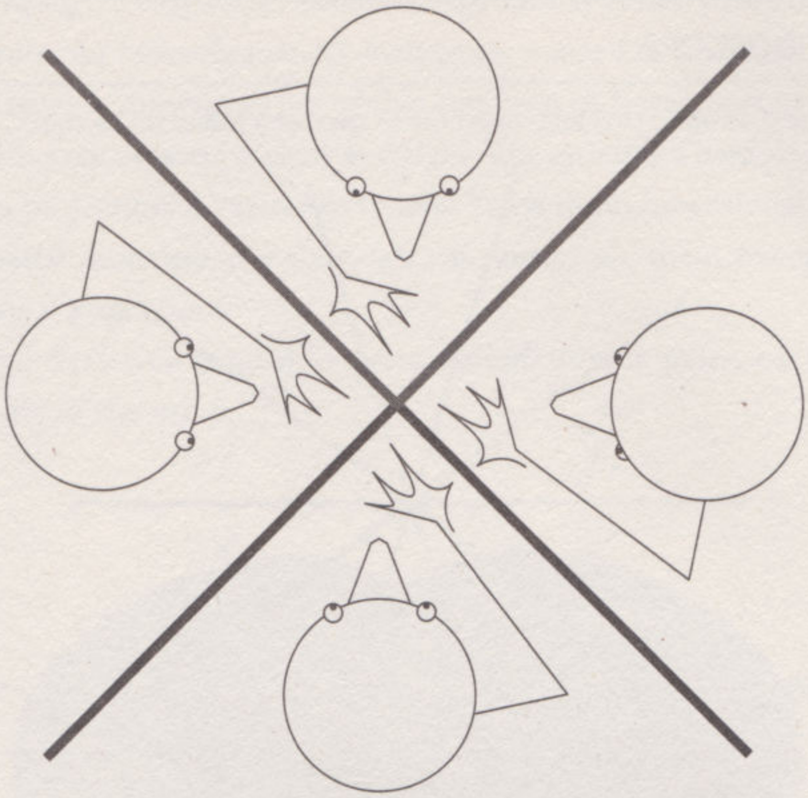


When two parallel specular symmetries are composed, one after the other, the result is a translation in the sense of a movement.

When the specular symmetries are not parallel, the symmetry planes intersect in straight lines *R*, with angle α, and the result is rotations in respect to *R*, with amplitude 2α.



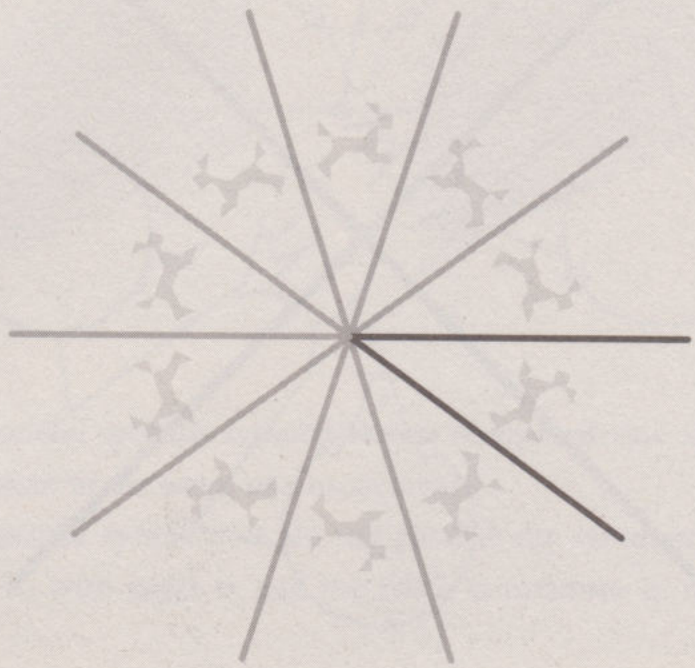
There is a curious characteristic of specular symmetry concerning mirrors – in actual fact, combinations of two mirrors – which do not invert the image, as is shown by looking straight into mirrors placed as shown in the illustration on the following page:



Games with specular symmetries have inspired such masterly scenes as those filmed by Orson Welles in The Lady from Shanghai.

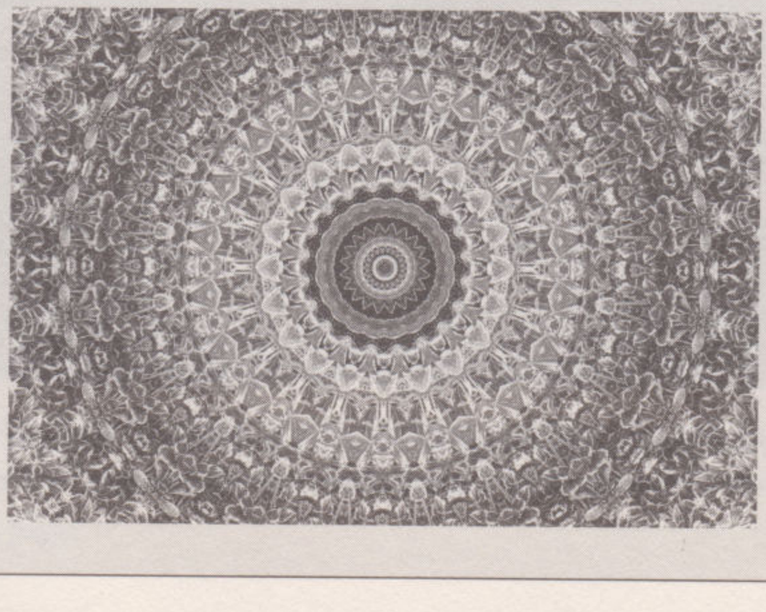
THE KALEIDOSCOPE

A kaleidoscope is an optical toy that makes use of symmetry planes for its effect.



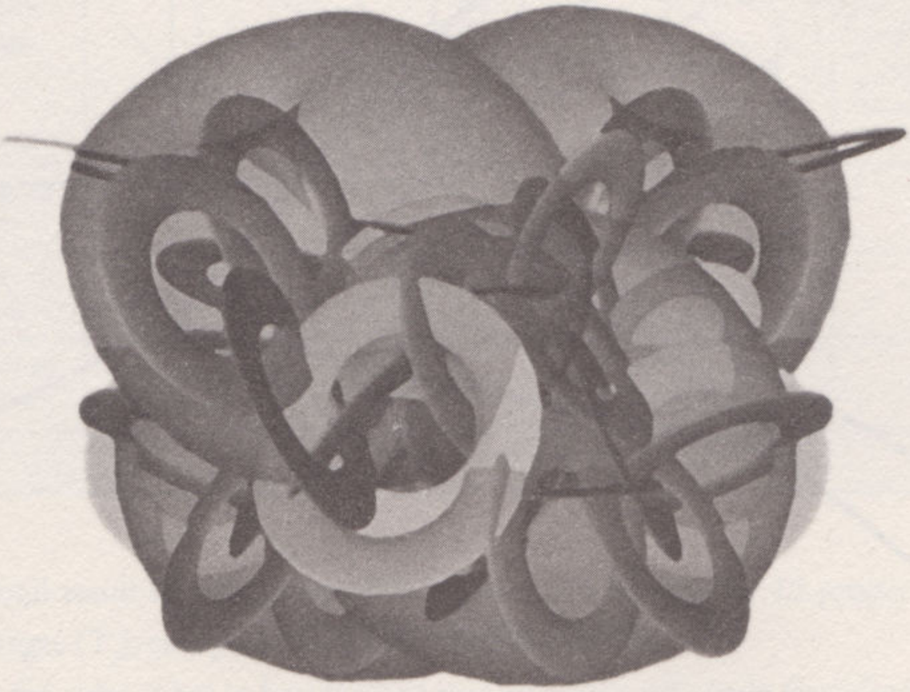
The symmetry planes that make a kaleidoscope work.

When two mirrors are placed at an angle of opening $\pi/3$ or $\pi/4$, the number of images produced of an object placed between them is 6 or 8 (as the image is also reflected in the two mirrors). Of course, the idea also works with amplitude π/n, which gives $2n$ images. And it has to be admitted, that for such a simple instrument, the kaleidoscope can produce some surprisingly complex results. Thanks to its specular symmetry and the nature of the objects reflected, it can create extraordinarily complicated and ever-changing images.



However strange it may seem, sometimes specular symmetry is difficult to detect. Delicate instruments of mathematical analysis are needed to find it, as the supposed specular symmetry is sometimes not visible, not even to researchers of string theory, who work with such esoteric objects as Calabi-Yau varieties. These do not represent a surface but a projection. A variety of Calabi-Yau is multidimensional – it has more than three spatial dimensions – although we cannot see them except in inferior two-dimensional projections.

Calabi-Yau varieties appear in pairs, called mirror pairs, each one being enantiomorphic of the other.

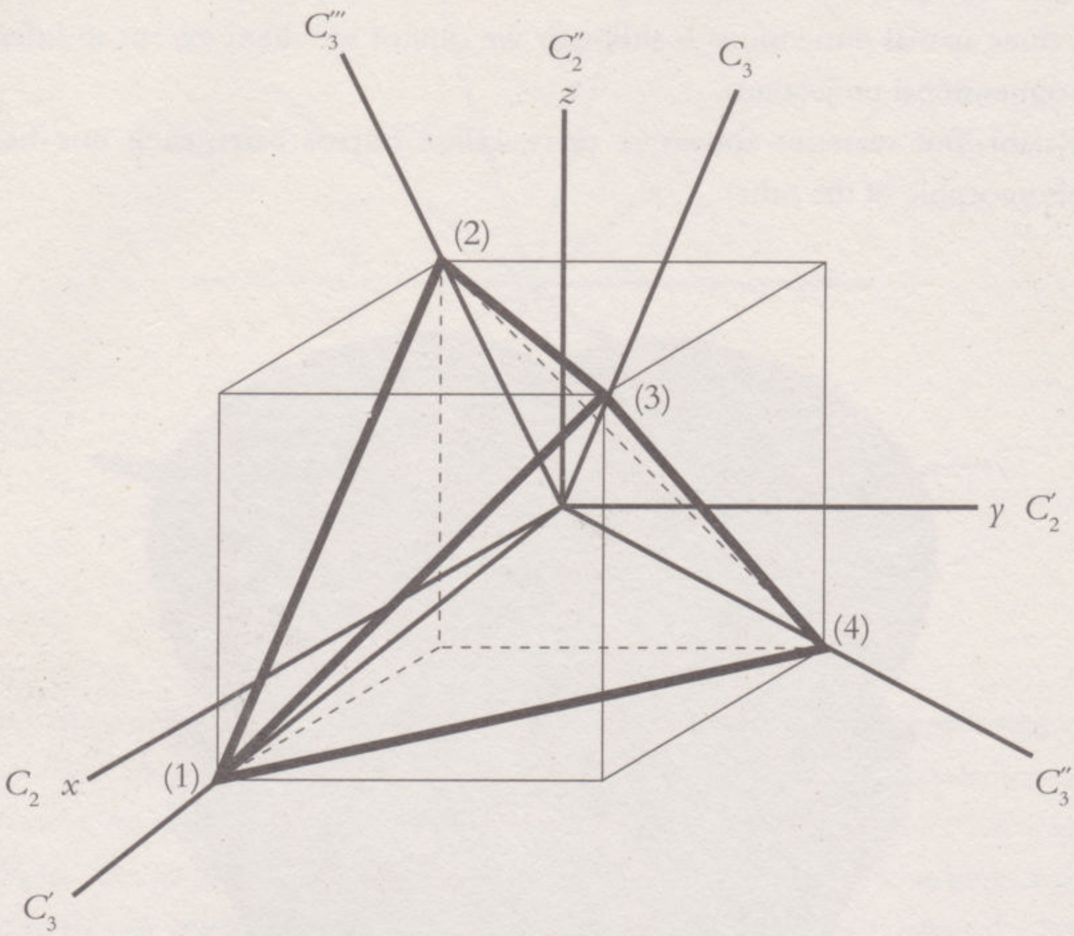


An example of a bidimensional projection of a variety of Calabi-Yau.

Some spatial examples

We have already seen the elemental symmetries of a regular polygon. Let's now look at those of a regular polyhedron. In principle, we should not find ourselves faced with any titanic task, as, though regular polygons are infinite, there are only five regular polyhedra, i.e. the so-called Platonic solids: the tetrahedron, the hexahedron (or cube), the octahedron, the dodecahedron and the icosahedron. Unfortunately, the symmetries of a polyhedron are not simple things and we soon discover that we are dealing with very large and complicated groups.

Let's begin with the tetrahedron, the simplest of the polyhedra as it has 4 vertices, 6 edges and 4 faces. Its group of symmetries will be as numerous as the permutations among its vertices. The tetrahedron has no fewer than 24 symmetries. It has several axes around which to turn, though some are difficult to find.



Rotation axes of a tetrahedron.

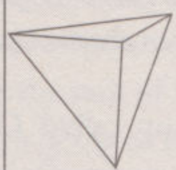
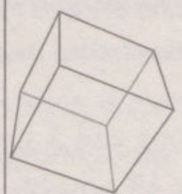
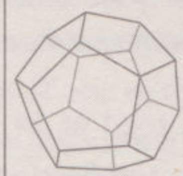
Rotations around the axes (there are 4) that pass through the vertices of the tetrahedron are of amplitude $2\pi/3$, and the rotations round the axes that pass through the median points of each edge (there are 3) are of amplitude π. All in all, it comes to a group of 12 proper rotations, which is the name given in geometry to rotations round an axis.

To find all the symmetries – 24 in total – we have to take into account what are called improper movements: 6 reflections on a plane (isometries resulting from being reflected in a mirror) and 6 rotoreflections (isometries that correspond to being reflected in a mirror and simultaneously rotating around an axis perpendicular to the plane).

PLATONIC SOLIDS

Platonic solids are regular convex polyhedra in 3-D. They get their name from their appearance in the Platonic dialogue *Timaeus*, in which they are also bestowed with magical properties. Fire, for example, was said to be composed of tetrahedra; air, of octahedra; water, of icosahedra; the earth, of hexahedra; and the air, of dodecahedra.

At every vertex, the same number of edges coincide, all with the same length and faces forming regular polygons. The geometric characteristics are summed up in the table below.

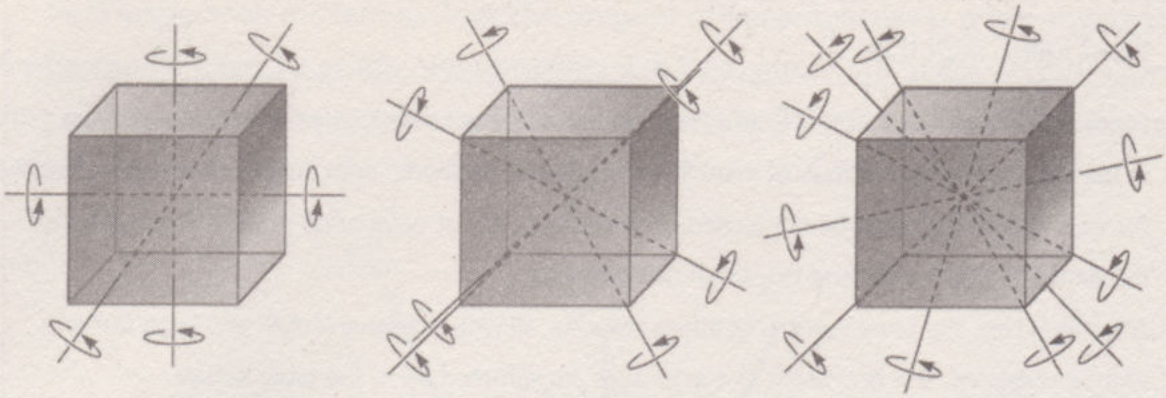
Platonic solids	Tetrahedron	Hexahedron	Octahedron	Dodecahedron	Icosahedron
					
Number of faces	4	6	8	12	20
Number of edges	6	12	12	30	30
Number of vertices	4	8	6	20	12

The total result is two groups $T \subset T_d$, one (T) formed by 12 proper rotations and the larger (T_d), composed of all the symmetries: 24.

As for the cube, it has to be admitted that one starts off a little fearfully bearing in mind how difficult it was to cope with the tetrahedron. But that fear is unfounded as it is easier to visualise a cube.

Its geometric structure coincides with the mathematical concept of the three spatial directions. These are better known as the three coordinate axes to which we are well accustomed from secondary-school maths.

The isometries of a cube are proper rotations and their complement, improper; the proper rotations are rotations of amplitude $2\pi/3$ around the axes (4) which pass through the vertices, rotations of amplitude $\pi/2$ around the axes (3) which pass through the centres of the faces, and, lastly, rotations of amplitude π around the axes (6) which pass through the median points of each edge.

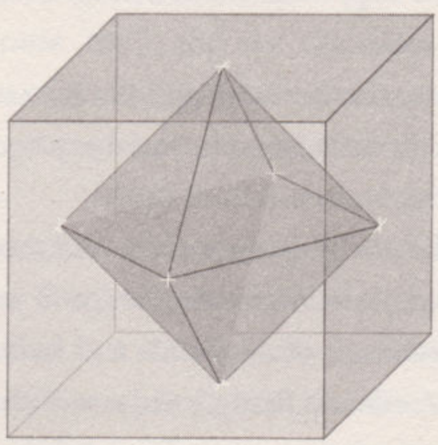


From left to right, the proper rotations of the cube: Three rotations around the axes that pass through the centres of the faces; four rotations round the axes that pass through the vertices; and six rotations around the axes which pass through the median points of each edge.

That gives a total of 24 proper rotations and a group called O. By including reflections and rotoreflections, it comes to 48 possible symmetries plus a group of considerable size designated O_h, which verifies

$$O \subset O_h.$$

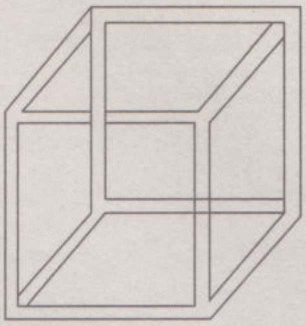
It would appear that things get complicated on moving on to the polyhedron that comes next. The octahedron seems even larger and it is to be expected that it will have more symmetries than a cube. But, no, it doesn't. In the three-dimensional realm, things function their own way, and it turns out that the cube and the octahedron are dual. This means, in simple terms, that a cube and an octahedron are linked by a surprising geometric fact, which is that by joining the median points of the faces of a cube, the result is an octahedron, and vice-versa. One consequence of the duality is that both figures have exactly the same symmetry group.



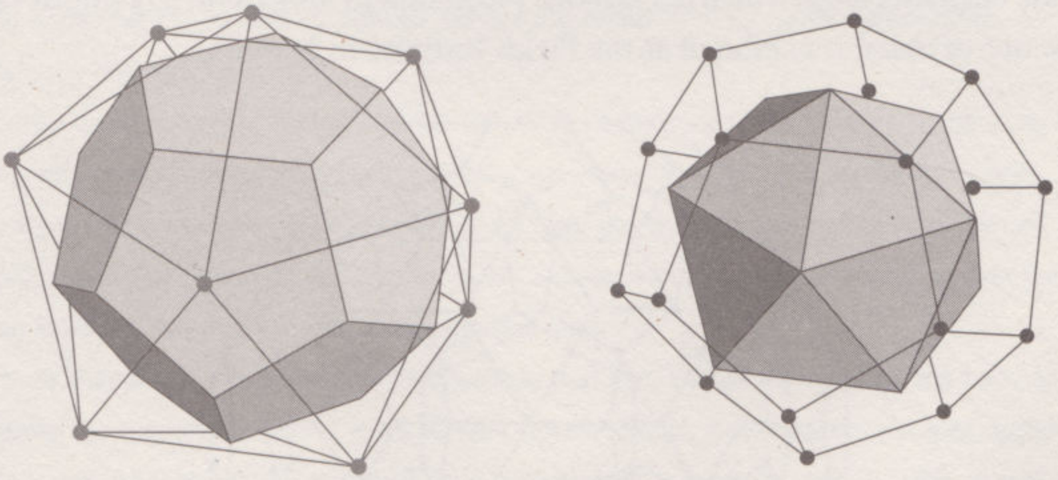
In symmetry, the octahedron and the cube are said to be dual.

A CUBE WITH NO SYMMETRIES

There is an asymmetrical variety of hexahedron: it is the one known as the Escher cube, but it is an impossible object in our three-dimensional Euclidean world. In one of this Dutch artist's works entitled *Belvedere*, we can see this cube surrounded by other impossible perspectives.



The situation with the little monsters that come next is somewhat bitter-sweet as, while the dodecahedron and icosahedron luckily turn out to be dual as well, that does not save us the task of finding the symmetry group of one of them. But, not to worry; we are not going to respond to the provocation by calculating the symmetries. We will be content with knowing that the group has 60 proper rotations and 120 elements. In case you want to do the research on your own behalf, we'll give you a clue: the number of rotation axes of the icosahedron, for instance, is 10 of amplitude $2\pi/3$, 6 of amplitude $2\pi/5$ and 15 of amplitude π.



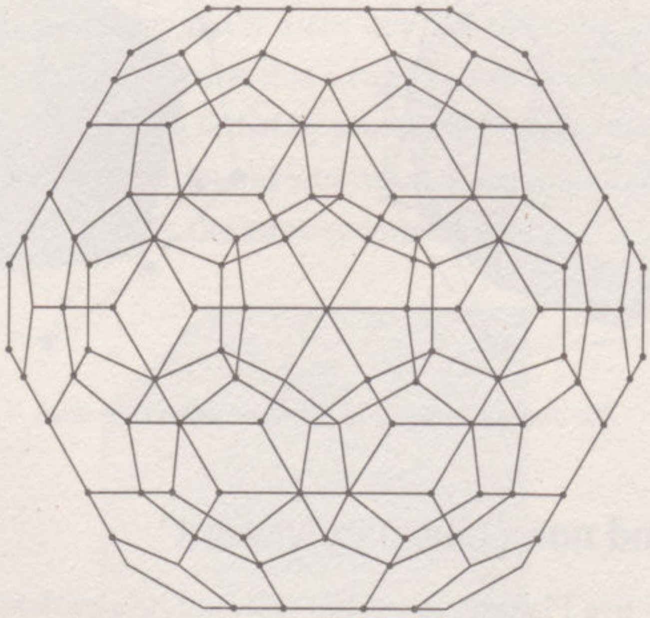
The dodecahedron and the icosahedron are dual.

Non-regular and non-convex polyhedra

There may only be five Platonic solids, but it would be a mistake to think that they complete the study of the spatial symmetry groups. There are many more varieties of polyhedra besides the regular ones, although it is true that on losing equal angles and sides some symmetry is lost, and therefore the study of other polyhedra in some way impoverishes the search.

Platonic solids are regular polyhedra. But they are only a small part of the bodies that geometers call polyhedral; there are semi-regular polyhedra, Kepler-Poinsot polyhedra, truncated ones, star polyhedra, Catalan polyhedra, acoptic ones, Johnson polyhedra, rigid, convex or flexible ones, with such exotic names as, rhombicosidodecahedron, cuboctahedron, rombicocuboctahedron or tetradecahedron. They all present symmetry, some of them in vast numbers, and their symmetry groups can be colossal. There have been blessed intellects, like that of H.S.M. Coxeter (see panel, opposite), which have moved around in this universe with ease, an ease that seems exasperating to the rest of us with a more limited capacity for visualising geometry.

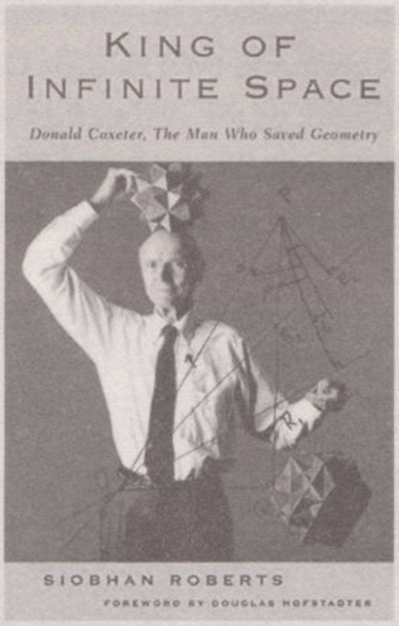
So unlimited was Coxeter's capacity to 'see', that he even worked with polyhedra in dimensions greater than three, so-called polytopes. For example, he dealt with symmetries of the 120-cell, a regular tetra-dimensional polytope that goes by the name of hyperdodecahedron, which has 600 vertices, 1,200 edges, 720 pentagonal faces, and 120 dodecahedrical cells. A four-dimensional polyhedron, as well as having vertices, edges and faces, has constitutive elements of dimension 3, in this case, dodecahedral 'cells'. Its appearance, projected on to two-dimensional paper, looks like the diagram below, which is a zenithal projection in dimension 2 (a similar figure made out of metal is exhibited at the Fields Institute in Toronto).



One of the main difficulties that specialists come up against is the definition of the term 'polyhedron'. Surprisingly there is no clear idea of what a polyhedron is. There is no universal consensus and the only thing on which everyone agrees (speaking of three dimensions) is that two faces delimit one common edge.

HAROLD SCOTT MACDONALD COXETER (1907–2003)

It is rare that figures such as the Anglo-Canadian H.S.M. Coxeter ever become world famous. Nevertheless, in scientific circles they are quasi-mythical beings, shamans of a way of how the world is, behaves, and can be understood. Coxeter wrote his first article at the age of 16, but was unable to submit his last one himself; it was published posthumously. His was a devotion that lasted almost 80 years and became practically a love story dedicated to groups.



Cover of King of Infinite Space, *the biography of H.S.M. Coxeter written by Siobhan Roberts and published in 2006.*

Coxeter's long life gave him the opportunity to meet such notable people as Bertrand Russell, Ludwig Wittgenstein and M.C. Escher. To give you an idea of what Coxeter was like, when the mathematician Asia Ivic informed him that she was going to be away for a short time as she about to give birth, Coxeter handed her a thick manuscript saying that it was for her to read to prevent her getting bored when she had nothing to do!

Coxeter could have been nicknamed 'Mr Group' as he dedicated his whole life to the study of symmetry and symmetry groups in Euclidean space. However, he did it in several dimensions, as the three that we perceive were not enough for him, and so he passionately devoted himself to multidimensional polytopes and polyhedra. His approach to this universe was more geometric than algebraic, which made him something of a rarity.

Groups that admit a presentation on (meaning that they can be defined "on the basis of") specular symmetries are named, in his honour, Coxeter groups. Coxeter spent decades studying and classifying these groups, culminating in classification of the finite ones in 1935. As well as in polytopes, they appear in graph theory, in crystallography, in tasks involving Lie groups, in the theory of *buildings* (which has nothing to do with actual edifices) and others.

Coxeter wrote 13 books, one of which is a compilation entitled *Kaleidoscopes*, which was a topic that he always found fascinating and on which he made some very profound observations. He was also an accomplished musician and in the prologue to his *Regular Simple Polytopes* he wrote: "I have made an attempt to construct the book like a Bruckner symphony with crescendos, little foretastes of pleasure to come, and abundant cross-references." His concept of symmetry was like a work of art, too.

If we limit ourselves to the polyhedra traditionally dealt with, and keep to the properties relevant in symmetry, polyhedra can be isogonal, isotoxal, isohedral, regular, quasiregular, semi-regular, uniform and noble. Their symmetry groups can be enormous, even though Coxeter and others have classified them into families. Just think of the symmetries of a simple prism with a regular cross-section, which admits as many rotations as faces. It is enough just to know that no fewer than 10 families can be distinguished by studying a polyhedron's symmetry groups.

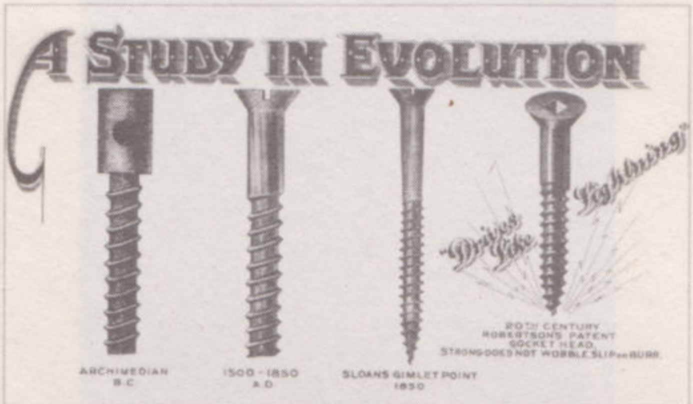
JACQUES TITS (b. 1930)

Jacques Tits, a Belgium-born, naturalised French mathematician, is a super-specialist in group theory, an exceptional algebraist, was a close friend of H.S.M. Coxeter and is a proponent of Coxeter's ideas and research. His contribution to the knowledge of groups is vast and his surname associated with the group known as the Tits group, a simple finite group of order 17,971,200. He is the creator of the concept of *buildings*, combinatory elements linked to groups, and of many other mathematical ideas. He holds the honour of being a member of the Bourbaki group, mathematicians who collaborated under the name Nicolas Bourbaki.

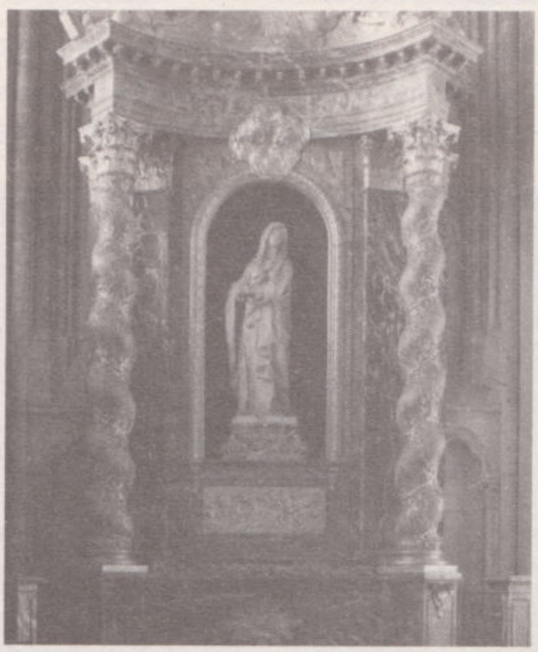
Jacques Tits (above left) with (in the background) the mathematician John Griggs Thompson receiving the 2008 Abel Prize from Norway's King Harald V.

Helicoidal symmetry

A helix is a tridimensional mathematical curve characterised by one basic property: The angle forming its tangent with a determined straight line (known as the axis of the helix) is always the same. To refer to the surface that joins the helix to its axis we give it the name helicoid. If the helix is wound round a cylinder, the resulting object is symmetrical and familiar to everyone as it is to be found on, and in, columns, drills, screws and a multitude of other objects and places.

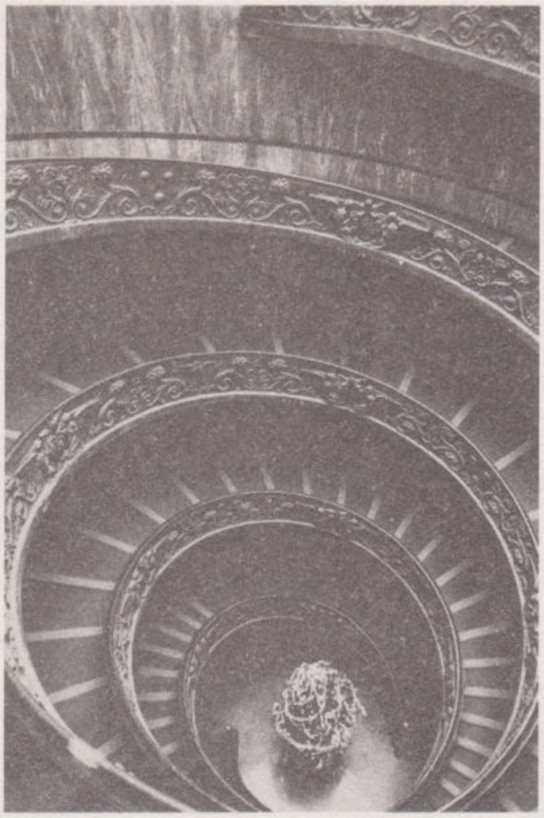


The thread of a screw is a good example of helicoidal symmetry. The illustration is a 1909 advertisement promoting products from the Robertson company.

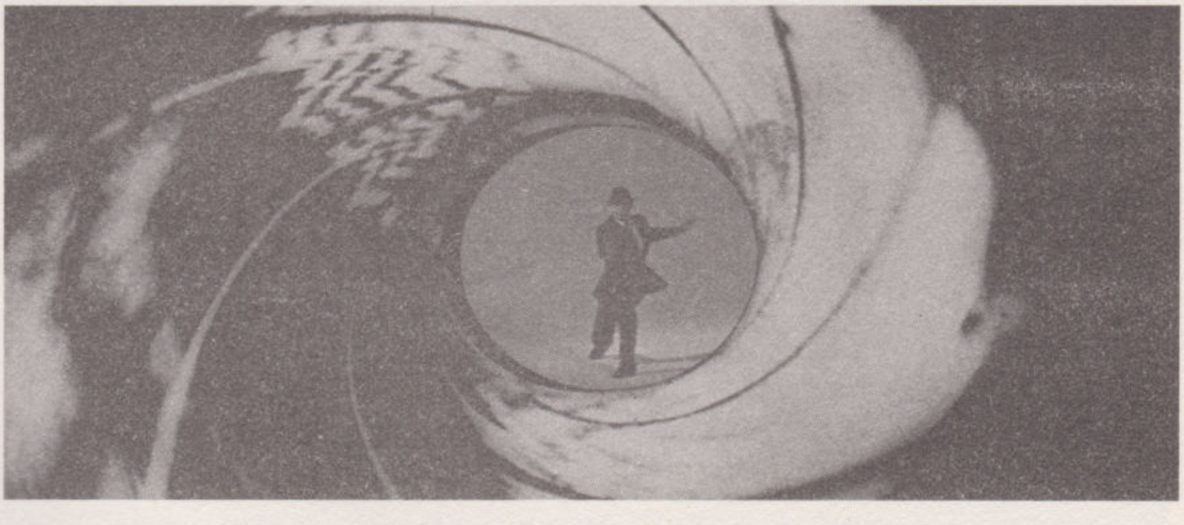


Solomonic (helical) columns at the altar of the Virgin of Mercy, Notre Dame Cathedral, Amiens, France.

Helices are chiral, in other words, they can be helices wound to the right or helices wound to the left, and therefore their enantimorphic images do not coincide. Sometimes two enantimorphic helices are made to turn around the same axis, with the result being some curious structures.



A double stairway in the Vatican, Rome. One staircase is for going up, and the other, wound round the first one in a helix, is for going down. Another outstanding example of a double staircase can be seen at the Château de Chambord in France.



A universal icon: a title sequence from a James Bond movie showing the hero framed in the helicoidal rifling of his enemy's gun barrel.

A bolt and its nut, is a very everyday example of two enantiomorphous varieties. One of the helix's important elements is its pitch, which determines its symmetry range. Basically, helicoidal symmetry is a tridimensional combination of translations in accordance with an axis and rotations of a certain angle. In ballistics, helices are put to use as the rifling of the barrels of pistols, guns and other firearms. Rifling causes the bullets and shells to spin, which ensures they maintain their trajectory.

Chapter 4

Groups and Equations

Go to heaven for the climate, go to hell for the company.

Mark Twain

Bad company – very bad indeed, considering it cost him his life – was the undoing of Évariste Galois, just before he reached the age of 21. But let's not get ahead of events, as his full story will be told in detail later in this chapter. Galois was a mathematician and an enthusiast of equations. And, though it might not be obvious at first glance, equations are in the origins of symmetry groups. A first-degree equation consists of finding the zero of a first degree polynomial. The solution to the equation is the value of x, which we should say now is called the unknown variable, that makes the equation true.

If the equation is

$$ax + b = 0,$$

the only solution is

$$x = -b/a.$$

When the equation is second degree, the polynomial is also second degree, and takes the form

$$ax^2 + bx + c = 0.$$

Both solutions have been known almost since ancient times, and among the Greeks (who understood them as magnitudes of a geometric type), Arabs and Mediaeval Europeans, they eventually became the formula studied today in schools,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

with x_1 and x_2 being the solutions sought. We are, in fact, speaking – very much ahead of time – about the fundamental theorem of algebra, proven by Jean-Robert Argand (1768–1822) in 1806 and in its present form by Carl Friedrich Gauss (1777 –1855) in 1849, which states that a polynomial of real coefficients of degree n has n roots, real or complex. Expressed in mathematical terms, every polynomial $P(x)$ of degree n, $P(x) \in \mathbb{R}|x|$, has n roots in $\mathbb{C}$.

Non-elementary equations

If the polynomial is third degree, the Italian Scipione del Ferro (1465–1526) found an arithmetical formula which when elaborated by subsequent algebraists, takes us to the 'simple' (as only some would say) expression

$$x_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}} - \frac{1}{3a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}}$$

$$x_2 = -\frac{b}{3a} + \frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}} + \frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}}$$

$$x_3 = -\frac{b}{3a} + \frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}} + \frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}}{2}}$$

for the three roots of $P(x) = ax^3 + bx^2 + cx + d$, which demonstrates his outstanding intellect. Del Ferro, of course, did not write it in this fashion (it even has components with the symbol i, which refers to the imaginary unit of complex numbers, something still from Del Ferro's intellectual tool box). His version was more elementary, and limited to third-degree equations of a particular form, but it is easy to put it all into general modern language. Let's remember that in his time they did not have the advantages of present-day notation, which would not really appear until the times of Viète and Descartes. In those days they still operated with formulae written in conventional literary language, with a few abbreviations to simplify making repetitions.

Del Ferro's feat pales beside the formula for solving the fourth-degree equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

which was also found by an Italian, Lodovico Ferrari (1522–1565) in 1540, at only 18 years of age! The arithmetical formula for finding the four eventual solutions is truly fearful, if not monstrous, however much modern notation is used. What Ferrari did, in modern terms, was a change of variables, introducing $X = x + b/4a$ and with that he eliminated the third-degree term. He next carried out some subtle operations which led to the solution equivalent to a cubic or third-degree equation. He thus reduced a very difficult question to one that was simply difficult, a technique that is most effective for solving any problem. We must add, however, that Ferrari's precociousness and ingenuity would not be enough to prevent him from losing his life in his early forties, poisoned with arsenic by his sister.

What about fifth-degree equations? Once the previous ones had been solved, there was nothing to indicate that in time a solution would not be found. That is, a formula in additions, subtractions, multiplications, divisions and roots. A miracle formula; no doubt complicated, but basically calculable. But the formula never arrived. Let's go over this first story of equations in rather more detail.

The story of Tartaglia and Cardano

Niccolò Fontana, known as Tartaglia (c.1500–1557), is the first of our characters. He was very lucky to reach adulthood as, when he was 12, a French soldier sliced across his throat during the siege of Brescia – in which his father was killed – and only a miracle saved his life. His speech must have been affected as his nickname can by translated into English as 'Nicolas the Stammerer'. He had a very difficult child-

hood of abject poverty, and it is known that he more or less taught himself to read and write. He earned his living as a mathematician, but was always on the lookout for a better-paid occupation.

Tartaglia, as he appears in one of his books, published in 1546.

At that time, public contests were the order of the day. Challenges were issued and authentic bloodless battles were promoted and attended, on occasions, by large crowds, including members of the nobility. First prize ranged from a salary raise to a banquet with the city's leaders. Tartaglia is said to have won 30 of these feasts in a row. One famous challenge pitted him against Antonio Maria Fior, Del Ferro's right-hand man. On his death bed, Del Ferro gave Fior the secrets of his procedures for solving third-degree equations, those with the form $x^3 + mx = n$. Some time later, Fior challenged Tartaglia to a maths battle with third degree equations, and was convinced that his formula would give him the victory. But Tartaglia, a very skilful algebraist, had already grappled third-degree equations, and was able to work out the equations in his own way under pressure – including those that did not follow the pattern of Del Ferro's equations – and he beat Fior with ease.

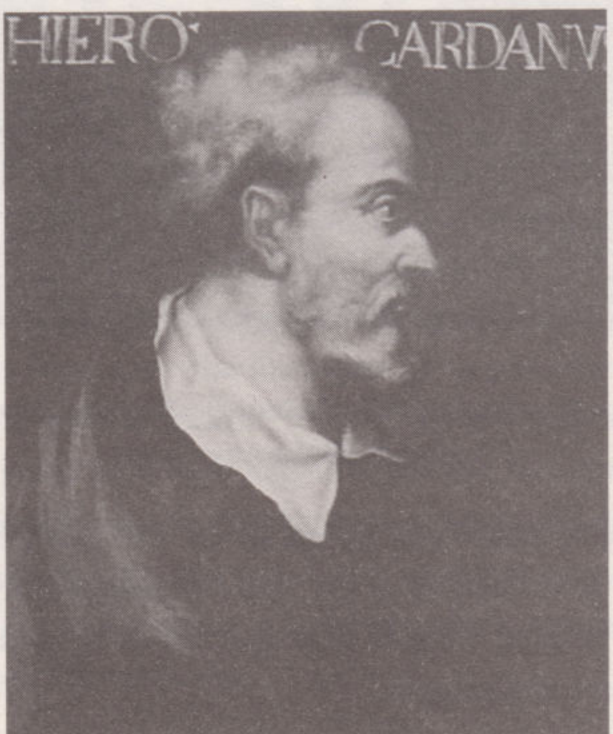
Tartaglia also made notable contributions in the field of ballistics, a discipline that could be classified as applied mathematics. Though today we might describe Tartaglia's contributions as somewhat basic, he was the first to determine that projectiles do not follow straight trajectories but wholly curved ones. (Later, Galileo demonstrated that the curves were, in fact, parabolas.) Tartaglia also showed that the faster the speed at which the projectiles were fired, the less curved was the trajectory. He also showed that the maximum range was obtained by firing at an angle of $\pi/4$ radians (45°, as Tartaglia did not count in radians).

A page from Nova Scientia, *the book in which Niccolò Tartaglia set down his discoveries in ballistics.*

Finally, it should be noted that his name was also given to Tartaglia's triangle (overleaf), which is better known as the Pascal or binomial triangle. This is an algebraic expression, possibly Chinese in origin, which in modern times has found many applications in combinatorics.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1

The man destined to be Tartaglia's bitter enemy was called Gerolamo Cardano (1501–1576) who had studied medicine at university and for whom mathematics was a hobby. Cardano had been born out of wedlock and in his days illegitimate sons did not benefit, except in exceptional cases, from the same privileged treatment as legitimate sons. In financial terms, this meant that Cardano had to fend for himself. While he was a student, he coped quite well as he was undoubtedly



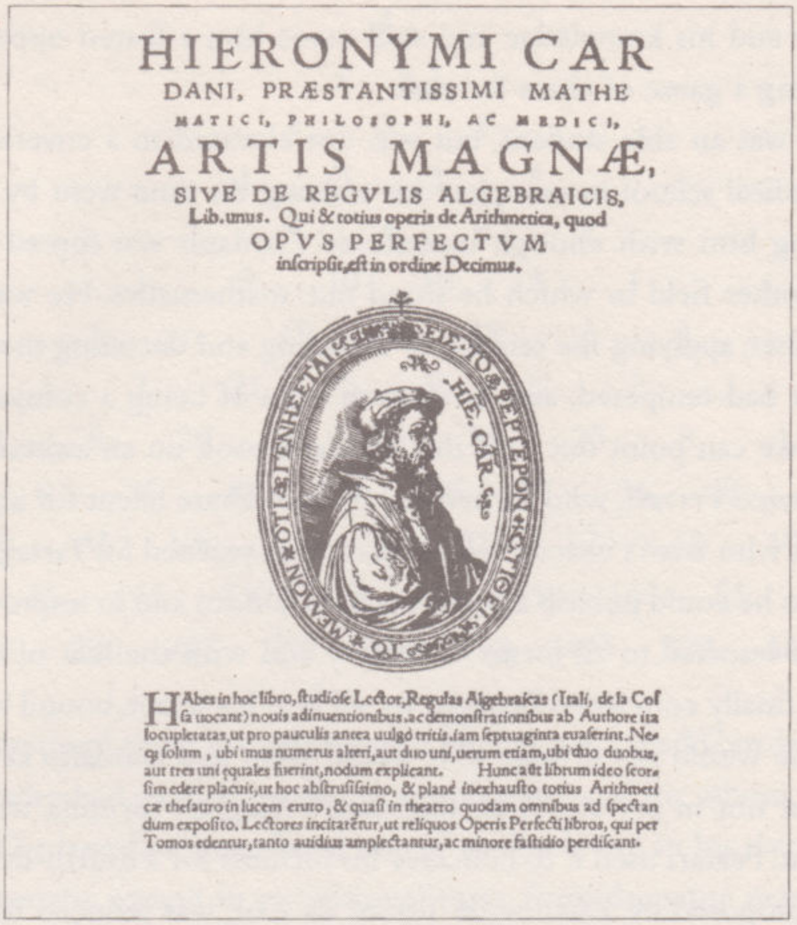
A portrait of Gerolamo Cardano, the mathematician who was in bitter dispute with Tartaglia.

a bright boy and his knowledge and skill made him a feared opponent at cards, dice or playing a game of chess for cash.

Cardano was an able student, but was not elevated to a coveted place at the Milanese medical school to complete his training. As time went by, gambling was not providing him with enough income, so Cardano was forced to devote his efforts to another field in which he stood out: mathematics. He was employed as a cryptographer, applying the science of encoding and decoding messages. He was always rather bad-tempered, and he showed signs of being a compulsive gambler. As an aside, we can point out here that Cardano took on an assistant, a boy of 14 named Ludovico Ferrari, who turned out to have a rare talent for algebra himself.

Cardano, who wasn't exactly rolling in money, yearned for Tartaglia's breadth of knowledge so he could publish a book, become famous and so improve his financial situation. He resorted to all forms of flattery and with the bait of offering him a better salary, finally convinced Tartaglia to tell him his secret, bound by the promise, that is, that he would not divulge it. It would seem that Cardano kept his promise in public, but not in private, as he discussed Tartaglia's formula with Ferrari, his sharp assistant. Ferrari used it to conceive his formula for a fourth-degree equation. A problem proposed by Zuanne de Tonini da Coi, was reduced to the equation $x^4 + 6x^2 + 36 = 60x$, which Ferrari then managed to solve.

Enter Annibale Della Nave, Del Ferro's son-in-law, who had inherited a manuscript containing his father-in-law's mathematical achievements and showed it to Cardano. Cardano needed no further excuse to break his promise to Tartaglia and saw this development as the route to fame that he craved: he no longer needed Tartaglia, as Del Ferro's alternative formula had reached his hands through other channels! He considered himself free of any oath he had made and published all his discoveries – including Ferrari's on the fourth-degree equation – in *Ars Magna* (original title *Artis Magnae Sive de Regulis Algebraicis*), a book that became famous in 1545. The solutions published by Cardano were not more useful than, for example, the algorithms of approximation of Jamshīd al-Kāshī (c.1380–1429), but they were authentic solutions, not approximations.



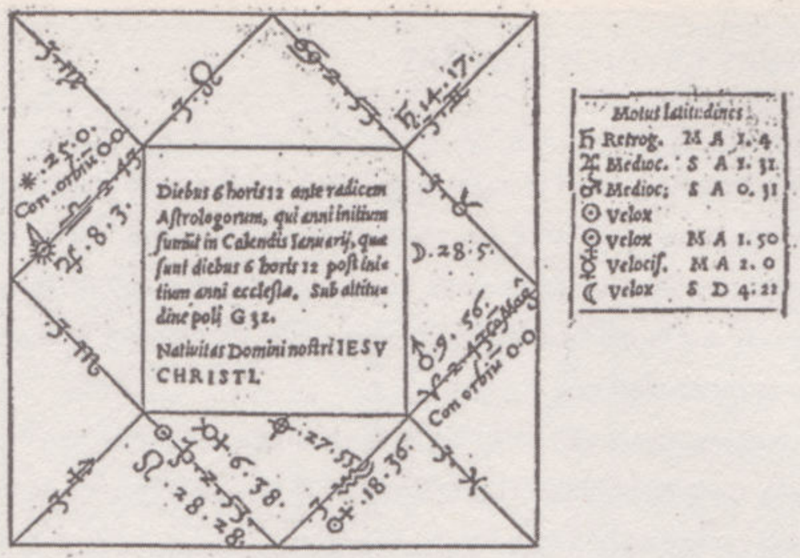
HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aût librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarentur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

The cover of Cardano's Ars Magna,
his famous treatise on algebra published in 1545.

Thirsty for blood and convinced that Cardano had betrayed him and disclosed an inviolable secret, Tartaglia decided that Cardano deserved to be challenged to a public contest. Cardano was intelligent enough to keep quiet and avoid the confrontation, and it was his assistant Ferrari, clever and ambitious, who stood up and responded with a number of challenges designed to throw Tartaglia's algebraic skills into question. Tartaglia was obliged to respond and, in 1548, a public contest was arranged in Milan. After only 24 hours of competition, Tartaglia acknowledged his inevitable defeat, abandoning the contest and the city.

From all this turbulent affair there would emerge only one winner: mathematics. Tartaglia died nine years later without having managed to improve his financial situation. Cardano, even though he became a doctor of great prestige, fell into disgrace due principally to his sons, who caused him countless problems – the elder was executed while the younger one was sent into exile. Cardono fell into the hands of the Inquisition; he was accused of heresy for drawing up a horoscope of noone less than Jesus Christ himself!



Reproduction of Cardano's horoscope of Christ.

Nevertheless towards the end of Cardano's life, things looked up a little and he even managed to draw a papal pension. According to the stories, Cardano brought his work on horoscopes to a climax by preparing one for himself in which he even set the date for his death. When the date went by and Cardano found himself still alive, he apparently felt obliged to honour his horoscope and took his own life.

Ferrari did not have better luck. Although his confrontation with Tartaglia had resulted in his employment as a tax inspector – and, naturally, in his becoming rich – he was murdered at the age of 42. The culprit is believed to have been Ferrari's sister, who inherited his estate and immediately transferred all rights to his work to her loving husband. The husband, without more ado, left her.

A fruitful interregnum

A fourth-degree equation is relatively easy to solve. It is not that the present-day fairly monstrous formula – elaborating on Ferrari's – is simple to calculate, but rather that, however complicated its calculation might be in practice, it is, after all, no more than a question of adding, subtracting, multiplying, dividing and extracting roots. In other words, it requires little more than diligently applying the four tools of mathematics.

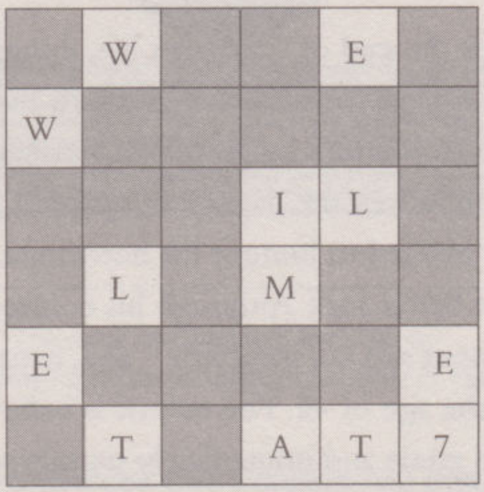
A fourth-degree equation can be solved by applying a formula. Other equations, truly colossal ones of much higher degrees than four, were later solved throughout the 20th century. In fact, Viète solved this equation posed by Adriaan van Roomen (1561–1615) in 1594:

$$x^{45} - 45x^{43} + 945x^{41} - \ldots - 3{,}975x^3 + 45x = 0.$$

16TH CENTURY CRYPTOGRAPHERS

For cryptographers, the century in which Viète and Cardano lived, the 16th, can be considered a blessed one. The astuteness of François Viète (1540–1603) caused Philip II of Spain to believe that his adversary, Henry IV of France, had made a pact with the Devil. The French seemed able to read his most intimate thoughts and discover every command he issued. In reality, Viète was in the service of Henry IV and he did no more than decipher Philip II's messages, a possibility which simply did not occur to the Spanish king.

The story of Gerolamo Cardano (1501–1576) is less dramatic than Viète's, but he was also involved in cryptography. Cardano invented a cardboard grille in the shape of a square or a rectangle. The holes permitted the writing of a message. For example:



If the message were long, the grille could be turned (up to three times as long as letters were not superimposed). When the grille was removed, the paper beneath looked like this:

	W			E	
W					
			I	L	
	L		M		
E					E
	T		A	T	7

The empty spaces were then filled with random letters and numbers and the message was sent looking like this:

5	W	1	S	E	9
W	T	4	K	2	E
H	C	U	I	L	5
8	L	R	M	Z	7
E	Q	S	8	0	E
G	T	B	A	T	7

The recipient put his copy of the grille in place to decipher the message.

	W			E	
W					
			I	L	
	L		M		
E					E
	T		A	T	7

Who could extract the real message if they didn't have the key to the cypher, that is, the grille? Nowadays not much time would be needed to decipher such a wordsearch puzzle, but in Cardano's times it was not so easy. His method was practical, portable, reproducible and simple. In suitably modified form, the technique was still in use at the beginning of the 20th century.

But Viète did it without using any formula and the fact is that no one could find any formula, whether simple or not, to solve each and every equation of the fifth degree or above by using only elementary arithmetical rules. The fact that valid, simple formulae for solving all kinds of equation had not been found despite all the years gone by and the efforts of so many scholars now baffled the best mathematicians. Even Lagrange had begun to think that perhaps for some unknown reason, miraculous formulae simply did not exist.

As time went by, that belief grew and later it became a certainty: there is no miracle formula for solving quintic and higher equations. Those responsible for informing us of this impossibility were Paolo Ruffini and Niels Abel. The latter finally came up with unassailable proof. Though it was ingenious and mathematically impeccable, it did not open any doors; Abel's demonstration is an excellent example of reasoning, but it goes no further than what it says. Furthermore, there are fifth-degree equations that can, in fact, be solved by means of formulae that involve only elementary mathematical operations, including radication (taking the root), but that does not mean that a simple formula exists that works for all. Thanks to Abel, it was known that there are equations

PAOLO RUFFINI (1765–1822)

Secondary school pupils might remember Ruffini for Ruffini's rule, an elementary algorithm used when studying polynomials, but no doubt for nothing else. However, this Italian physician and algebraist was a prolific thinker with a razor-sharp mind. He was the first to seriously attempt to prove that no general formula existed for quintics, and also the first to use some of the resources of group theory to demonstrate it. Unfortunately, some of his writings were difficult to digest and, despite filling the desks of important mathematicians and societies with the seeds of his ideas, he was never given the recognition that he hoped for. In fact, the occasional gaps have been found in his demonstration, but he would perhaps have been able to correct these himself if anyone had carefully read – and understood – his explanations. In any case, Ruffini's contribution to algebra was not just in the study of quintics, and he is credited with many interesting results and concepts. Sadly, during an epidemic of typhus, he had the courage to care for all his patients, to become infected himself and to write up his own symptoms... and to die in less than a year.

which are not solvable, now or ever, but it was not known why some equations could be solved by formula and why others could not. The ultimate explanation was given by Galois, and our interest alights on his writings. We shall not reproduce Abel's demonstration here because it is not relevant to our purposes; it is Galois and symmetry that are under our spotlight.

NIELS HENRIK ABEL (1802–1829)

One of the most notable mathematicians of all time, Abel lived in poverty for most of his life. Born in Norway, his father was a theologian and his grandfather a Protestant clergyman, so it was no surprise that he took up the study of religion. However, at school Abel developed surprising mathematical skills, and his teacher, friend and benefactor, Holmboë, gave his full support to Abel's scientific vocation. With a modest scholarship grant from the Norwegian government, Abel set off to travel round Europe to meet mathematicians and visit institutions. Perhaps the most notable thing that happened during the trip was the links he established with August-Leopold Crelle, and the foundation of the *Journal de Crelle*, the first mathematical magazine ever to be published. In 1824, Abel proved that no general formula existed for fifth-degree equations. It is encouraging to point out that some years before, Abel himself believed he had proved the opposite, but he realised that his reasoning had been incorrect. News of the result, the work of a young, obscure Norwegian, does not appear to have spread far, and Abel's merits were not acknowledged until after his death, when the Paris Academy awarded him and Carl Jacobi (1804–51) the Grand Prix. Abel, meanwhile, had made substantial contributions to elliptic series and integrals, but fell victim to the tuberculosis which was to take him to his grave. He managed to see his fiancée for the last time and just get past Christmas, but his health immediately worsened just when his friend Crelle had managed, at last, to find him a paid position in Berlin.

The story of Galois

Évariste Galois (1811-1832) had all the qualities of a romantic hero, and that's where one of the keys to his popularity surely lies, as seen in the accounts of his life and works, both serious and sensational.

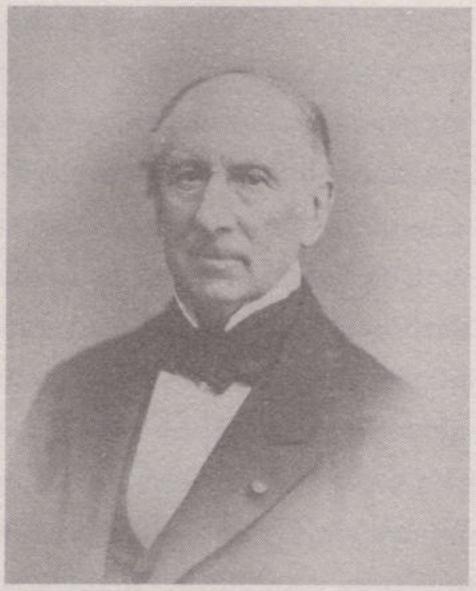
Galois was born into a family which could be called progressive, and as a child he had to cope with the suicide of his father. This was due in large part to the slanderous gossip of a local priest who had so much of a grudge against Galois' father that he dedicated to him some spiteful verses set to music regarding the morality of his wife. It was a rather odious affair, reminiscent of the argument of the opera *La Dolores*.

Galois grew up in a nonconformist environment, with his disinterest at school for some subjects being matched by his genius for mathematics. He was not an easy person to deal with. In her writings, the mathematician Sophie Germain (1776–1831), who came to know him briefly, does not speak highly of his character. She described it as despotic and offensive. The young Galois was a revolutionary activist, a passionate enemy of the Church, and an adversary of the Bourbon monarchy and Louis Philippe d'Orléans.

One of the two known contemporary portraits of Galois.

Galois' capacity for mathematics earned him the admiration and protection of his teacher, Louis-Paul Émile Richard. Many years later, this same man would become tutor to Charles Hermite (1822–1901); with Hermite he triumphed, but with Galois, he failed. Throughout his school life, Galois showed an aptitude which must have seemed almost like a challenge to his teachers, particularly in view of his defiant, arrogant attitude. Ignoring the advice of many, he took the entrance exam for the École Polytechnique at the age of 17, a year early. Galois was not admitted, but he got over that setback by writing a paper on algebraic equations and submitted it to the great Cauchy (see panel, below), who told him he would comment on it. That never happened – Cauchy is said to have lost Galois' paper. Another theory is that he asked Galois to develop and extend it, simply postponing its presentation to the Paris Academy, but the truth about this episode is that it is shrouded in mystery.

AUGUSTIN-LOUIS CAUCHY (1789–1857)



A French mathematician, one of the greatest ever seen, Cauchy had a personality which earned him a lot of enemies. The fact is that, perhaps due to the influence of his family and upbringing, he was an extraordinarily religious man, an admirer and supporter of the Jesuits, absolutist, and armed with unyielding convictions and beliefs. For many years, he preferred exile in Italy or Switzerland rather than betray his convictions by swearing loyalty to Louis Philippe d'Orléans or Napoleon III. When Napoleon fell and the returning absolutists carried out reprisals against people such as Gaspard Monge (1746–1818) and Lazare Carnot (1753–1823), Cauchy had no qualms about taking their posts. He lost out, one after the other, on almost all the academic positions he applied for, as his own colleagues voted for his adversaries, and he even managed to fall out with someone as generous as Joseph Liouville (see panel, page 99). He was tutor to the third son of the exiled Charles X, though the dedication he put into this task was met with little success. Despite everything, he was a great mathematician and his academic prestige has not suffered at all. Everyone accepts his intellectual capacity, evidenced by his numerous achievements. Suffice to say that his complete works, considered at the time to be of national interest and therefore state subsidised, fill 27 volumes!

Galois returned to his studies and his beloved mathematics, and for a second time he took the École Polytechnique entrance exam. This time it went even worse: one of his examiners was particularly inept and his questions, which were irrelevant, exasperated Galois. It all came to an end when the examinee threw a blackboard wiper at the examiner. As might be expected, the young man was rejected for a second time. It may well be that the incident was worsened by the fact that it coincided with the recent suicide of his father.

With the doors of the Polytechnique closed, Galois applied to the École Normale, where he was admitted on account of his splendid examination results. He summed up his studies on equations – with groups now included – in a paper intended to compete in the Paris Academy Grand Prix and, no doubt, to win it. The secretary of this institution was the eminent mathematician Joseph Fourier (1768-1830), who took the paper home to study it... and then died, dashing Galois' hopes again. We must remember that we are talking about a period when there were no photocopies, no typewriters, no word processors, and no methods of electronic storage.

And, while we are talking of the epoch and its customs, mention should be made of one aspect lost from modern times: the duel. It used to be considered that to refuse to join in a toast (involving some powerful spirit or other) was an indication of a lack of manliness. It was also considered – fortunately this is not the case now – a lack of manliness to refuse an offer of a duel, which could arise even among close friends for quite trivial reasons. Incidents of this type became so banal that they could be more likened to a game of Russian roulette rather than a valid means of defending one's honour. Galois, in whom many detected signs of growing paranoia, put the unexpected death of Fourier and the consequent loss of his paper down to a conspiracy among the scientific establishment caused by their reluctance to accept innovative ideas from unknown youngsters. Looked at more soberly, it all seems less dramatic, but seen from Galois' fiery, youthful and frequently very drunk point of view, there is a certain logic to it.

His political stance hardened, and he ended up spending a month in jail for a toast in which he issued a veiled threat to the king, Louis Philippe d'Orléans. As far as maths were concerned, the enforced reclusion would surely not have affected him too much, as his prodigious mind did not need a pen and paper to think. He was later to spend another eight months locked up for illegally wearing the uniform of the artillery corps. Galois had by now become a fervent revolutionary and activist.

Meanwhile, he presented another much more elaborated paper to Siméon-Denis Poisson (1781-1840). However Poisson spotted so many errors and gaps in reasoning

that he recommend the academy reject it and he asked Galois to rewrite it with fuller explanations.

At about this time, Galois fell in love and there has been much speculation about the object of his desire. Subsequent research identified her as Stéphanie Poterin du Motel, the daughter of a doctor. Galois had already been involved in the odd drunken brawl, but this time he found himself obliged to agree to a duel over an amorous issue the details of which are unknown. His adversary may have been Pescheux d'Herbinville, a colleague of Galois. Some writers have claimed that behind this duel was the long arm of Louis Philippe's secret police, but that possibility is now considered unlikely. The confrontation is considered to have been particularly absurd as the rules of the duel permitted only one of the weapons, chosen randomly, to be loaded.

In the final hours leading up to the duel, Galois feverishly devoted himself to writing to his friends, in particular to Auguste Chevalier. The long letter is full of

JOSEPH LIOUVILLE (1809–1882)

Though an excellent mathematician himself, Liouville has gone down in history as the man who finally discovered Galois. Perhaps for that reason, he has never really received the credit he deserves, with his other achievements being regarded with a touch of scorn.

Liouville was quite fortunate in his academic life as he achieved and retained two professorships in the hallowed Collège de France and the Faculty of Science, Paris. His activities outside his professorships covered politics and, particularly, the foundation and running of the *Journal des Mathématiques Pures et Appliquées (The Journal of Pure and Applied Mathematics)*, a magazine destined to become a prestigious publication, and through which the writings of Évariste Galois were made known. Liouville was brave enough to publish them and, what's more, to give them an enthusiastic and sincere welcome. Liouville himself carried out a lot of useful work in mechanical physics and a number of theories bear his surname, such as the Sturm-Liouville theory and the Liouville-Arnold theorem. His contributions to continued fractions and the theory of numbers were significant and have the advantage of being highly comprehensible. More specifically, he built the first transcendental number, expressed by

$$N=\sum_{n=1}^{\infty}\frac{1}{10^{n!}}=0{,}110001000000000000000001\ldots,$$

which belongs to those known as 'Liouville numbers', an infinite family.

annotations of the type "*Je n'ai pas le temps*" ("I haven't got time"). In the letter, he gave a summary of his mathematical theories, and issued instructions for his ideas to be communicated to Gauss or to Jacobi, two men who could judge them. Perhaps Galois knew he was going to die.

On May 30, 1832, the young Galois was shot in the stomach and was left bleeding on the field of honour until a farm worker found him and took him to hospital on a cart. There he was to die in the arms of those closest to him. When Galois saw their despair, he murmured to his brother, according to the friends who were comforting him, : "Cry not, I need to be brave to leave my life at the age of 21". Yet he was still only 20!

Chevalier did what the letter said, and Galois' ideas reached the world. For years the world ignored them. It was much later, in 1846, that Joseph Liouville dug them out from among the papers of the Academy, and, full of admiration, at last had them published with the respect they deserved in the *Journal de Mathématiques Pures et Appliquées*. By then, the world was ready to welcome them; the pity is that Galois by then was of this world no longer.

Galois' theory

Simplifying greatly, let's follow Galois' steps in pursuit of that elusive formula of quintic equations. This line of reasoning appears nowadays in many books and websites, but it is always a source of wonder that a young man of not even 21 should have worked it out. As often happens, the clearest way is to provide an example. Let's take the equation of rational coefficients – it is very important that for the moment we should limit ourselves to these coefficients:

$$x^4 - 46x^2 + 289 = 0,$$

which is fourth degree and has four solutions calculable by roots or radicals (in what is sometimes called a biquadratic equation), which are:

$$\begin{aligned} \alpha &= -\sqrt{3} + 2\sqrt{5} \\ \beta &= -\sqrt{3} - 2\sqrt{5} \\ \chi &= \sqrt{3} + 2\sqrt{5} \\ \delta &= \sqrt{3} - 2\sqrt{5}. \end{aligned}$$

If we consider the possible permutations among them, we will find that there are 24 possibilities. We know today that they form a group called S_4 of order 24, but this concept was unknown in Galois' times. The four solutions, of course, solve the polynomial

$$P(x) = x^4 - 46x^2 + 289,$$

but if the permutations of those roots are analysed, we see that there are subtle differences of symmetry among them. Paraphrasing George Orwell, some permutations are less equal than others. For instance, if we consider the following expressions, also of rational coefficients,

$$\alpha + \delta = 0$$
$$\beta + \chi = 0$$

we find that they are true. It can be seen that, from the point of view of *P(x)*, nothing changes if we go on to carry out the permutation of S_4

$$\begin{pmatrix} \alpha & \beta & \chi & \delta \\ \delta & \chi & \beta & \alpha \end{pmatrix}$$

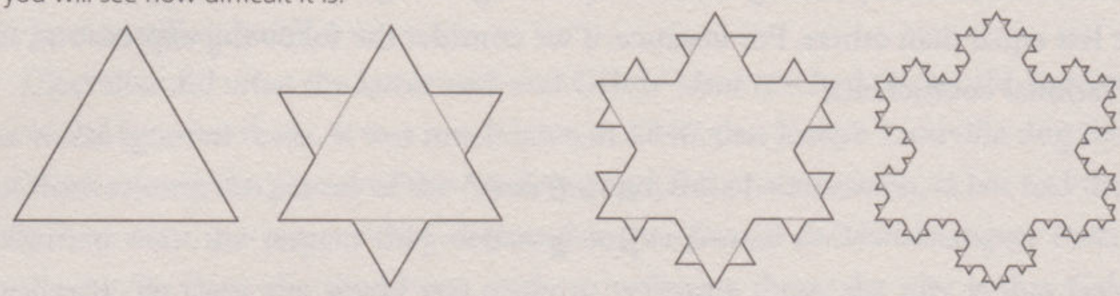
as the result is the equalities

$$\delta + \alpha = 0$$
$$\chi + \beta = 0$$

and nothing has changed. If the roots are interchanged, everything remains unaltered. But the fact is that any other algebraic equation satisfied by the said solutions, of radical coefficients, would also remain unaltered. Let's reiterate: any true algebraic expression for rational coefficients would also remain unalterable by a similar permutation of the roots, which makes the roots concerned indistinguishable. If the polynomial *P(x)* had a soul, it would live in ignorance of the permutation, as there would not be an expression of coefficients to denote it. This statement, which is not evident unless one reflects deeply on it, is something that Galois proved.

CAMILLE JORDAN (1838–1922)

Camille Jordan was a qualified engineer and earned his living in engineering for a large part of his life. He is mainly known in mathematics for the curve theorem that bears his name. It states that any plane curve that does not intersect itself (in other words, that does not entwine itself) divides the plane into two regions, "inside" and "outside" the curve, in such a way that if two points from each region are joined, the path that joins them intersects the curve. It seems elementary, but if you attempt to prove it except for non-differentiable curves, as, for example, a Koch curve, you will see how difficult it is.



The curve introduced by Helge von Koch (1870–1924) in 1904 is impossible to draw, even though it can be defined and imagined. The reason, among other things, is that it is a fractal object of dimension $d = \ln 4 / \ln 3 = 1.26186....$ The diagram reproduces the four first steps to the hypothetical aspect of such a curve. After an infinite number of repetitions, with more and more prickles, the curve can be obtained. It is continuous but not differentiable. It is better known as the 'snowflake curve'.

It is so difficult to satisfactorily prove something that is apparently so simple, that a few years ago it was still thought that Jordan's original demonstration was not entirely correct and that its reasoning contained some error or other. In reality it did not seem very clear to Jordan's contemporaries. Although Jordan made notable contributions to the theory of numbers and matrices calculation, he was also known for the boost he gave to the Galois theory – as soon as Liouville informed him of it – and the budding group theory. Jordan did not see groups as abstract entities as they are considered today, but as groups of permutations. That did not, however, prevent him from considering the chains of normal subgroups of the type

$$\{n\} = G_1 \subset G_2 \subset G_3 \subset ... \subset G_n = G$$

and from proving, together with Otto Hölder (1859-1937), the Jordan-Hölder theorem. This determines, with the exception of certain isomorphisms and subject to some quite elementary initial conditions being satisfied, the equivalence of such chains.

There are other algebraic expressions that do change on interchanging δ with α and χ with β:

$$\alpha-\delta=-2\sqrt{3}+4\sqrt{5}$$
$$\delta-\alpha=2\sqrt{3}-4\sqrt{5}$$
$$\beta-\chi=-2\sqrt{3}-4\sqrt{5}$$
$$\chi-\beta=2\sqrt{3}+4\sqrt{5},$$

but note that the expression's coefficients are not radical, so they do not form part of the permitted expressions. The 'good' permutations, those that do not change the algebraic expressions of rational coefficients, form a group, which today we call a Galois group. This is another conclusion – that they form a group – which is also difficult to demonstrate, and also due to Galois.

In this way, there are symmetries in S_4, or a symmetry group of the permutations of four elements, which form part of the Galois group, and others that don't. The permutation

$$\begin{pmatrix} \alpha & \beta & \chi & \delta \\ \chi & \beta & \alpha & \delta \end{pmatrix}$$

interchanges α and χ and does not lead to a true expression, as $\alpha + \delta = 0$ becomes $\chi + \delta = 2\sqrt{3} \neq 0$. Therefore, it does not belong to the group.

If we were to look for all the 'good' permutations, which would be quite a laborious task, we would find that there are only 4 among the 24 possible for S_4, that the Galois group of those 4 permutations in the end turns out to be isomorphic to the Klein group, which, in turn, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

In general terms, irrespective of the degree, every equation has its group, and that group is dependent on their roots and structure. But Galois went one step further; Galois' result says:

> An equation is soluble if, and only if, its Galois group is soluble, with a soluble equation being understood as that which can be solved by elementary operations and radicals.

What we understand as a soluble group was explained in the previous chapter.

The demonstration of this beautiful statement is not elementary and needs some 100 pages to be worked through calmly without rushing things. That is, at any rate, what it took Emil Artin (1898-1962), a great mathematician and author, who wrote *Galois Theory*, a book that changed maths history.

There are equations which, as a Galois group, have the fifth of the symmetry groups, S_5. To see if it is soluble it is necessary first to find the normal subgroups of S_5, which has no fewer than 120 permutations. However, after a lot of investigation, we find that only one chain of normal subgroups can be built, such as required by the supposed solubility (or, rather, solvability) of S_5:

$$\{n\} \subset A_5 \subset S_5$$

and on forming the required quotient groups, $S_5 / A_5 \cong Z_2$ and

$$A_5 / \{n\} \cong A_5,$$

it turns out, perversely, that in the end it isn't Abelian. Therefore, there exists no chain like those required by the fundamental theorem, so S_5 is not soluble, and, consequently, neither is the quintic equation. And what has been learnt about $n = 5$ is also valid for any successive value of n.

Although Artin's text is a classic, the theory known as the Galois theory is today applied in a more abstract and general way, because it is applicable to many other fields, much vaster than the simple consideration of equations. Nowadays rational coefficients are not postulated, and any body K, whether finite or not, is used. Nowadays we do not speak of expressions unaltered by a permutation, but of automorphisms α of K'/K, with K' being an extension of $K \subset K'$:

$$K' \xrightarrow{\alpha} K', \text{ with } \alpha(x) = x, \text{ for every } x \in K.$$

In its modern version, the Galois group is defined as the set of those automorphisms. Assimilation of it all is less natural, less intuitive, and is much more distant from the world to which we are accustomed, but it must be admitted that it gains by becoming almost all-encompassing.

The best praise that the modern version of the theory could be given is that no doubt Galois himself would love it, and would whisper to us: "Let no one forget: that's just what I said."

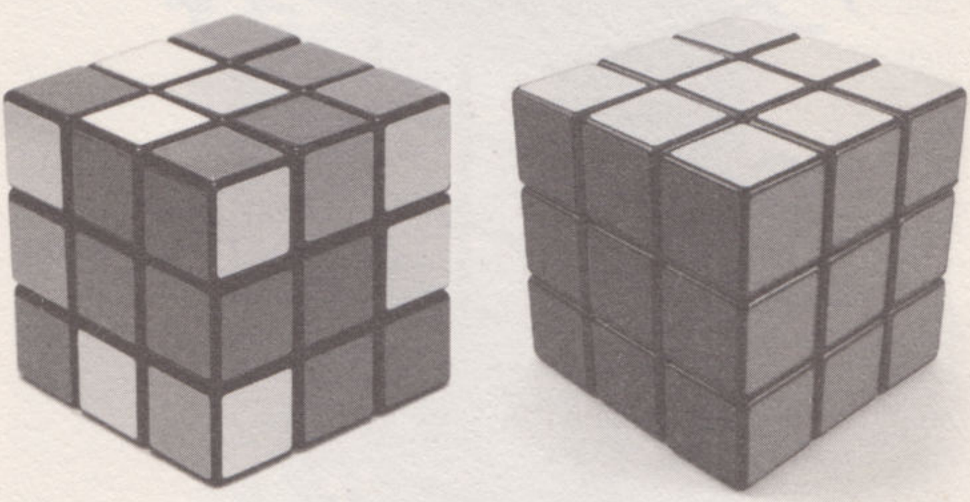
Chapter 5

Symmetry in Mathematics

Group theory is an exceedingly beautiful branch of pure mathematics used for showing in how many ways blocks of wood can be painted.

Robert Ainsley

If you are an enthusiast of mathematical games and puzzles, you come across groups all the time, whether you like shuffling cards or your interest is building up, taking down, or colouring wooden blocks. It is little wonder that groups are very much linked to the famous Rubik's Cube. The quote above, with its irreverent definition has some truth in it, though only partially. Group theory, one of mathematics' shining stars, a thing so beautiful that some new arrivals to its realm are overcome by it, can indeed be used to show the way to paint wooden blocks. But the sublime theory can also be used to solve a Rubik's Cube from any starting point.

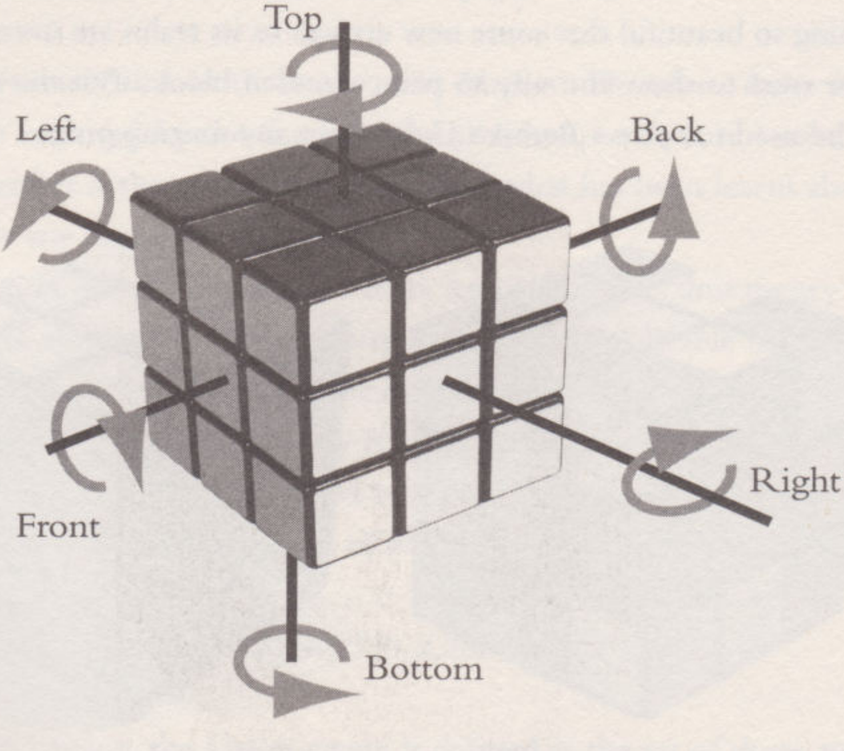


But solving puzzles doesn't seem a very worthy use for a whole mathematical theory. However, just have a try yourself at solving a random position of Rubik's Cube... It becomes a very different exercise, though, if you know that from any position you can go back to the initial position by correctly applying the rotations dictated by group theory.

Algebra for games

The biggest selling mathematical toy in history has had a long and successful career since it was created in 1974 by the Hungarian architect Erno Rubik. A lot of water has gone under the bridge since then, enough to give time for scholars to find quick solutions – though no easy ones – from any position that the small component cubes are in. There are 27 of these cubes, arranged in groups of three, and which, thanks to an ingenious rotation mechanism, can rotate in layers and change the colour of the faces shown. In its original version, the game consists of starting off from any position and attempting to get each of the large cube's faces showing a uniform colour. Taking it a stage further, different aims of the game can be invented – like placing another colour in the centre of each face, for example – but the game and the manoeuvres remain substantially the same.

Rubik's Cube allows a large number of movements in several directions forming a group of permutations of the edges and axes, complimented by the colours.



The rotations of Rubik's Cube can be reduced to those shown, applied to 1, 2 or 3 layers of small cubes.

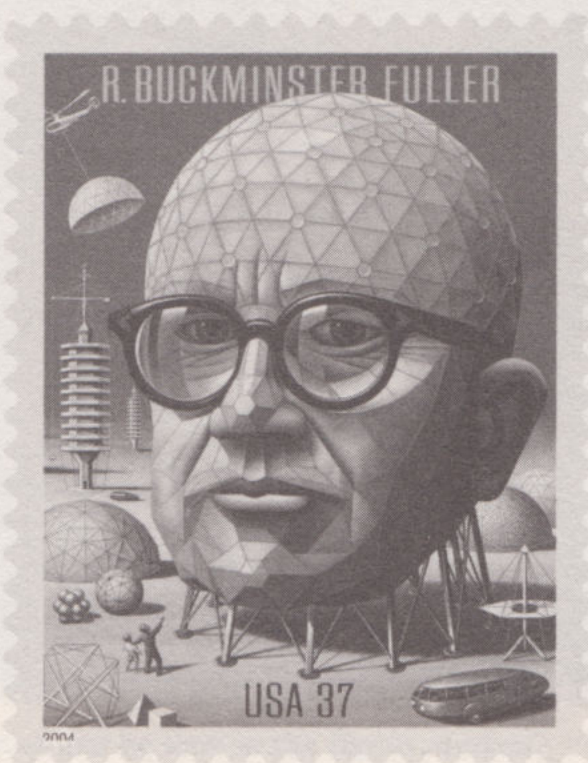
It's not an easy task to count up all the permutations. It is, in fact, a combinatory arithmetic exercise. All in all, there are 43,252,003,274,489,856,000 permutations which, luckily for the player, form a group. So everything is reduced to finding an algorithm that takes us, by means of a chain of movements, through to the desired

configuration. The fact that, with the actual size of the cube (about 6 cm along each edge), one cube for each of the cube's permutations would cover the planet nearly 300 times gives an idea of the puzzle's complexity.

The discovery of algorithms that give partial results has advanced greatly and there are thousands of papers, books and websites on the subject. It is not easy to master the best algorithms, but once you do, the time needed to find a solution decreases dramatically. Some people can complete the cube in less than a minute. In some cases, the puzzle has been solved in less than 10 seconds! There are all kinds of competitions and championships, though they are of more interest to the Guinness Book of Records than to algebraists. There are competitions for those who are blind, events where you have to find the solution holding your breath, and others where you have to manipulate the cube with your feet.

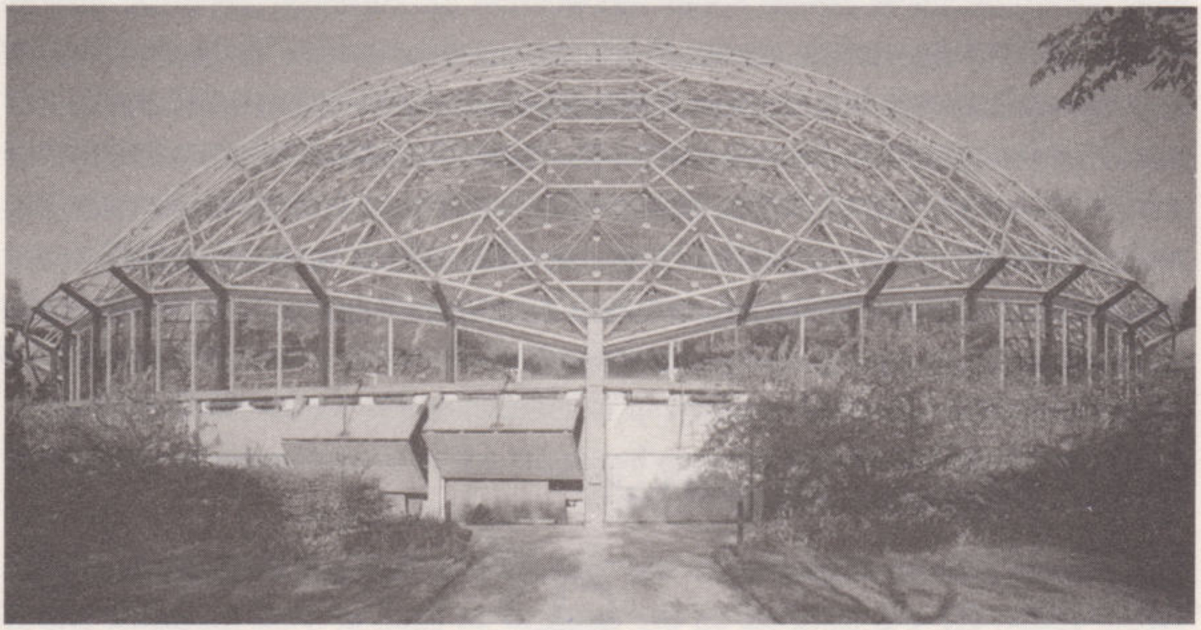
Domes, fullerenes and golf balls

One famous scholar of polyhedra was the architect Richard Buckminster Fuller (1895–1983). He was featured on the cover of an issue of *Time* magazine and is credited with introducing geodesic domes in building construction.



A commemorative stamp featuring Richard Buckminster Fuller, who, although he was not the inventor of geodesic domes, did develop their mathematical principles and patented them in 1951.

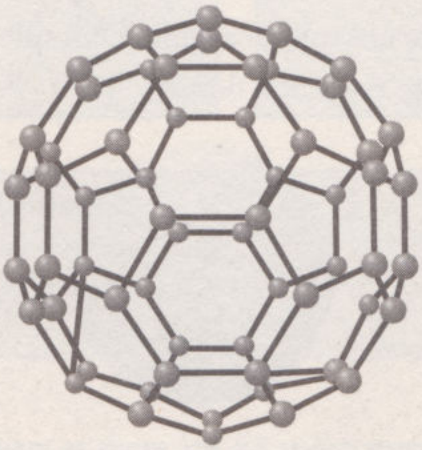
A geodesic dome is a structure that uses a network of polyhedra the faces of which, normally triangular, approximate a sphere when assembled together (as can be seen in the photographs shown below). Such structures have great strength, but weigh relatively little.



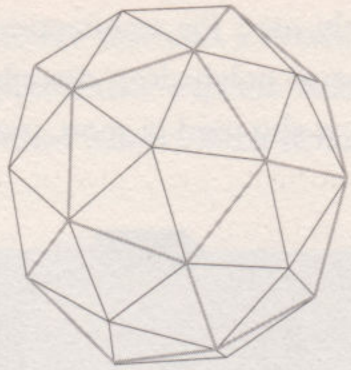
Exterior and interior views of the Climatron, a greenhouse built in 1960 for the Botanic Gardens of St Louis, USA. It is one of the many geodesic domes designed by Richard Buckminster Fuller.

These domes are not only used for architectural constructions. In fact, the truncated icosahedron, a type of polyhedron described as Archimedean and part of the basic theory of domes, is also used in sport, most notably to make footballs.

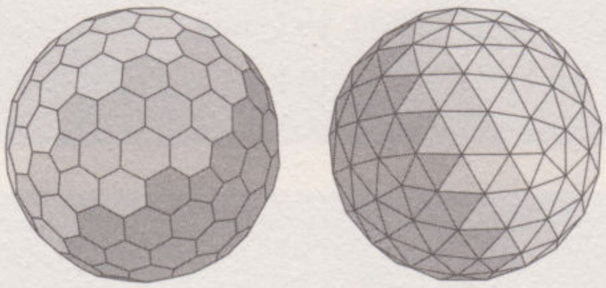
The truncated icosahedron is a polyhedron of symmetry group I_h (or $\star 532$, depending on the notation used) which recreates the C_{60} molecule, an allotropic form of carbon made up of 60 of the element's atoms. This variety was given the name 'buckminsterfullerene' (though it is now abbreviated to fullerene) by its discoverers, the chemists – and Nobel prize winners – Robert Curl, Harold Kroto and Richard Smalley:



The dual polyhedron that we are to discuss next is the pentakis dodecahedron, which looks like a dodecahedron that has grown a pentagonal pyramid on each face. However, it is formed of triangles that 'spike' the polyhedron and make it look more like a geodesic dome, as can be seen in the figure overleaf.



Duality is a regular occurrence in polyhedra, and consequently in the domes built out of them. A dual form is obtained by joining the median points of a dome with triangles – normally isosceles, not equilateral. Only the icosahedron – the one regular polyhedron dome has a dual with equilateral faces.



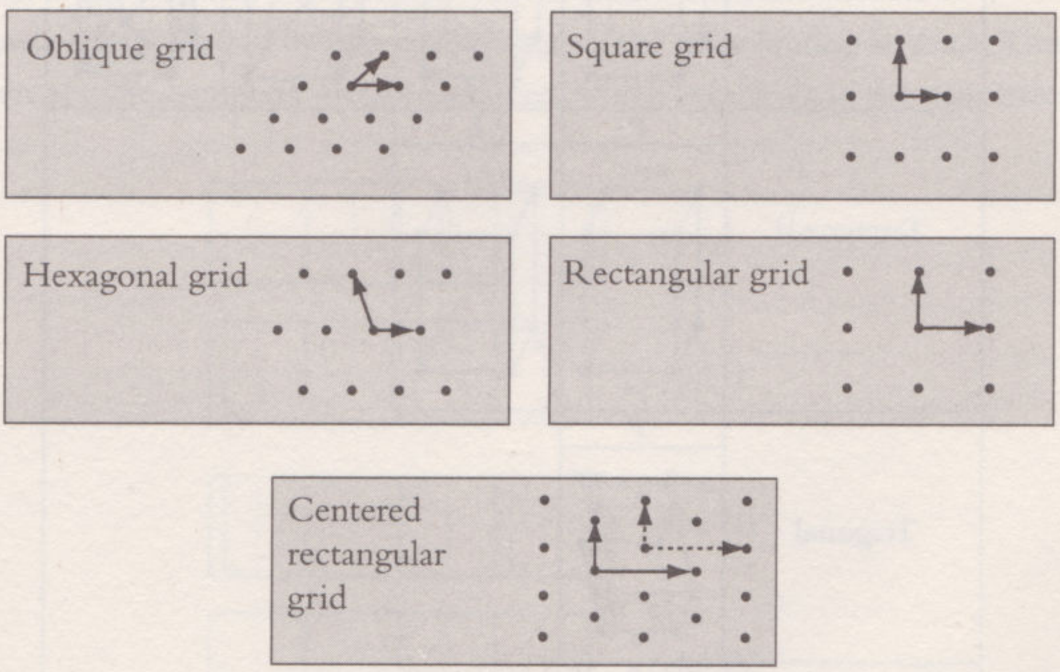
As for golf balls, they are not spherical globes but spheres with dimples and come under the geometric design rules of a dome.

Golf balls are designed to travel true and as far as possible. This means they must have the correct symmetry. On top of those demands there are also those regarding their manufacture, as the balls are made out of hemispherical pieces, and the pieces must not cut across the dimples. There is a plethora of literature available on such an irrelevant issue – irrelevant, that is, for non-golfers – and also an answer for those who are interested in the symmetry groups of a golf ball: It has between 48 – the octahedron group – and 120 elements.

Lattices

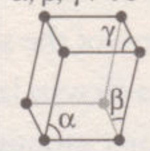
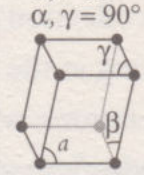
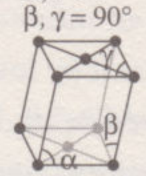
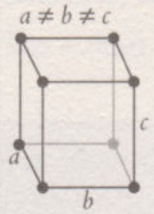
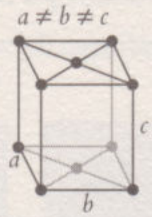
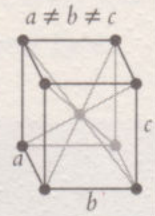
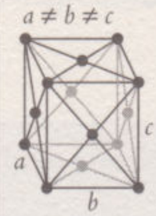
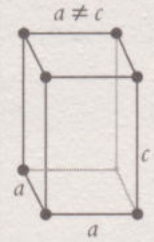
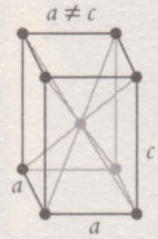
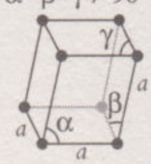
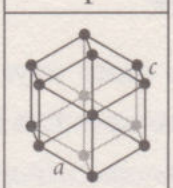
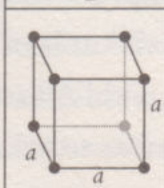
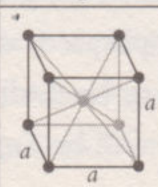
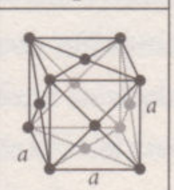
Point groups are those whose movements leave at least one fixed point on the plane. They govern the symmetry of centred bodies, which, like the polyhedra and polygons, have a privileged point, normally the centre of the shape. Named in honour of crystallographist Auguste Bravais (1811–1863), a Bravais lattice allows us to pass from the world of point groups to that of the non-point groups, that is, groups that do not affect just a single figure, but the whole of a plane or all space.

This type of lattice consists of an infinite collection of discrete points that are invariant through translations. It is as if by looking at the scenery from the inside of the lattice, the horizon were repeated in all directions, whichever way you look. In two-dimensional geometry, that leads to five configurations, described by vectors and angles.



There are no more lattices, as there are no more ways to cover all the plane. We shan't get into a detailed discussion here on this topic. It is not difficult, but it is rather laborious and depends only on the sum of the concurrent angles in a vertex. There are, then, five plane lattices, from which the groups affecting all the plane can be built.

In dimension 3, things get complicated, as all kinds of restrictions and cases that give identical configurations have to be dealt with. All in all, the following Bravais lattices result. Their names, which will be familiar to scholars of geology, are listed in the left-hand column overleaf:

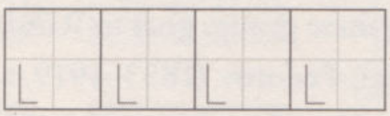
Crystal system	Bravais lattice			
	P	C	I	F
Triclinic	$\alpha, \beta, \gamma \neq 90°$ 			
Monoclinic	$\alpha, \neq 90°$ $\alpha, \gamma = 90°$ 	$\alpha, \neq 90°$ $\beta, \gamma = 90°$ 		
Orthorhombic	$a \neq b \neq c$ 	$a \neq b \neq c$ 	$a \neq b \neq c$ 	$a \neq b \neq c$ 
Tetragonal	$a \neq c$ 		$a \neq c$ 	
Trigonal	$\alpha=\beta=\gamma \neq 90°$ 			
Hexagonal				
Cubic				

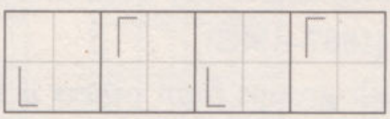
The sub-types P, F, I and C in the table correspond to the different types of the "face centring" and cross linking (whether belonging to the lattice or not) between the centres or the vertices The cross links can be imagined by putting the previous cells next to each other, so they touch vertices.

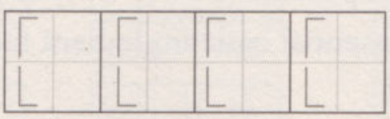
Wallpaper or mosaics, friezes and ornaments

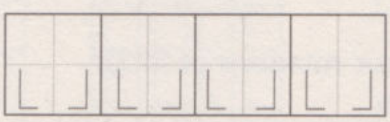
In reality, to speak of wallpaper, mosaics, friezes and ornaments is to speak of almost the same thing: of unlimited plane symmetries. Wallpaper or mosaics are expected to cover the plane, whereas friezes are limited by their edges. Ornamentation can be interpreted as friezes *interruptus* applied to other objects that are finite and in common use: hat bands, necklaces, pottery and so on. They are two sides of the same coin. As for symmetry, we can take it for granted, although we will ignore the cases in which it doesn't exist, as happens in a lot of ornamentation.

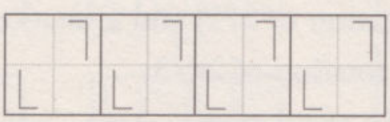
Mosaic friezes limited by two parallels are on the same footing as bands. There are seven possible symmetry groups for friezes, which are shown in the table below.

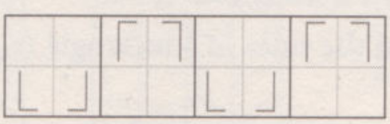

type I

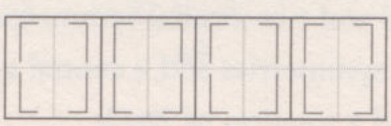

type II


type III


type IV


type V


type VI


type VII

In the Democratic Republic of the Congo, the Bakuba people attach great value to friezes and ornaments, particularly complicated and symmetrical ones.

A Bakuba design.

As for friezes, they can be deliberately made complicated and made to adopt the shape of columns; in that case there are exactly 24 groups, though we won't go into their description here.

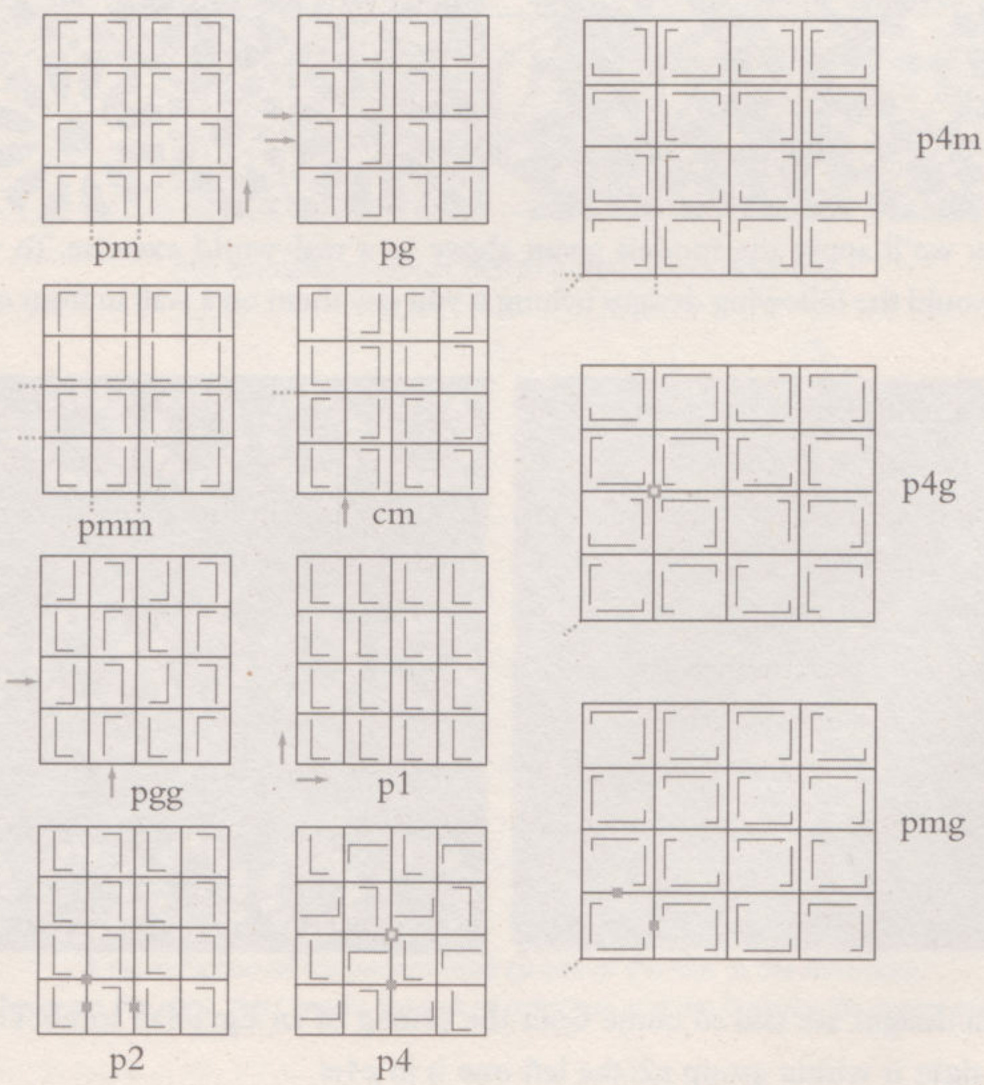
In dimension 2, the wallpaper or mosaic group – i.e. those that affect all the plane – can be of 17 types by combining the five dimension-2 lattices with the point groups. The credit for enumerating these groups goes to Russian mathematician and crystallographer Evgraf Stepanovich Fedorov (1853–1919, a revolutionary leader who died of hunger and tuberculosis) and to the work independently carried out many years later by George Polya (1887–1985).

The method used to distinguish groups from others is known as Hermann-Mauguin symbols and is rather complicated. Don't worry if you don't understand it very well at first, as it is not an essential basic ingredient for understanding what follows:

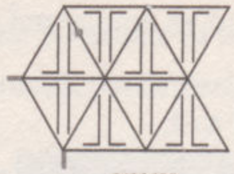
· *p*1: two translations;
· *p*2: three central symmetries (or rotations of $\pi/2$);
· *p*3: two rotations of $2\pi/3$;
· *p*4: a central symmetry (or rotation of $\pi/2$) and a rotation of $\pi/4$;
· *p*6: a central symmetry and a rotation of $2\pi/3$;
· *pm*: two axial symmetries and a translation;
· *pmm*: four axial symmetries on the sides of a rectangle (e.g. two horizontal and two vertical);
· *pmg*: one axial symmetry axial and two central symmetries;
· *cmm*: two perpendicular axial symmetries and a central symmetry;
· *p*31*m*: one axial symmetry and a rotation of $2\pi/3$;

- *p3m1*: three axial symmetries on the sides of an equilateral triangle (amplitudes of $\pi/3$);
- *p4g*: one axial symmetry and a rotation of $\pi/4$;
- *p4m*: three axial symmetries on the sides of a triangle, of amplitudes $\pi/4$, $\pi/4$, $\pi/2$);
- *p6m*: three axial symmetries on the sides of a triangle, of amplitudes $\pi/6$, $\pi/3$, $\pi/2$;
- *cm*: an axial symmetry and a symmetry with perpendicular sliding;
- *pg*: two parallel symmetries with sliding;
- *pgg*: two perpendicular symmetries with sliding.

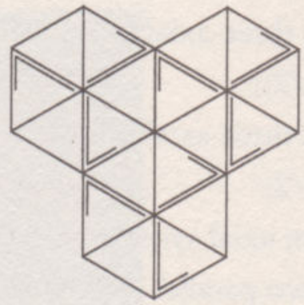
Jesús M. Landart's drawings make this a bit clearer. The points denote the rotation centres, and the arrows and dashes denote translations.



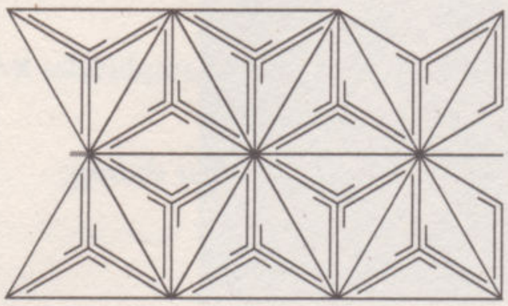
p31m



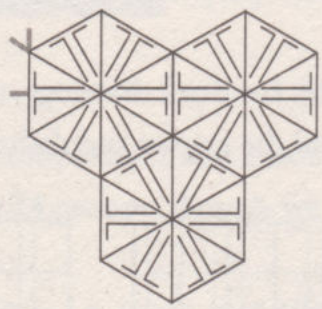
cmm



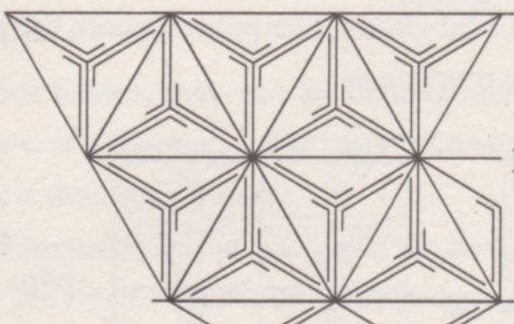
p3



p3ml

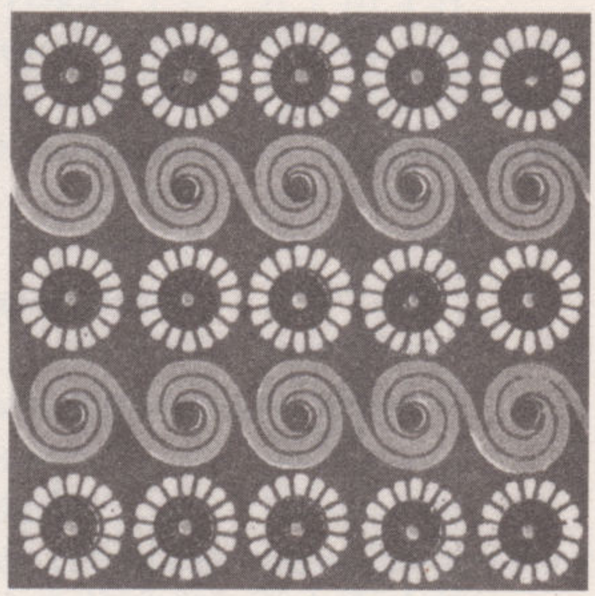


p6m



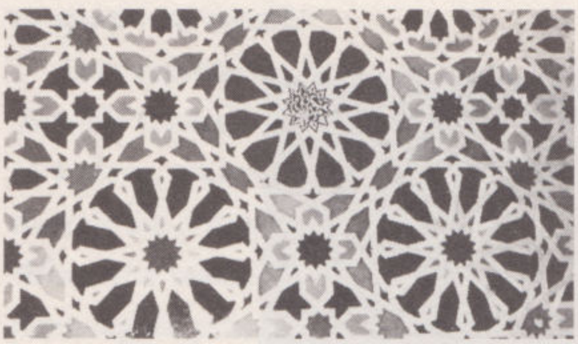
p6

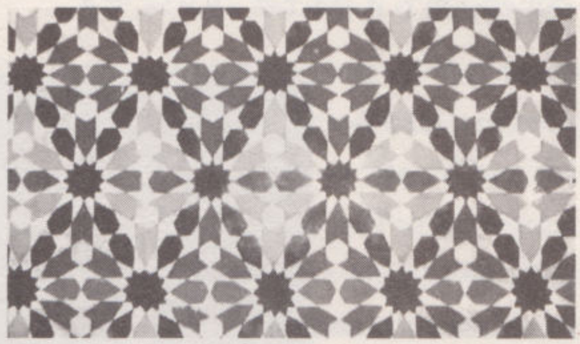
Now we'll apply the models given above to a real-world exercise. To which group would the following designs belong if you saw them on a wall in front of you?



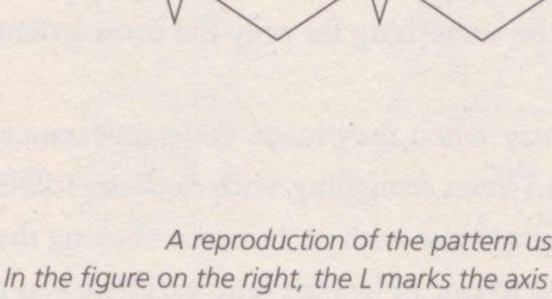
Both designs are said to come from the ceiling of an Egyptian tomb. The one on the right is within group *p*2; the left one is in *p*4*m*.

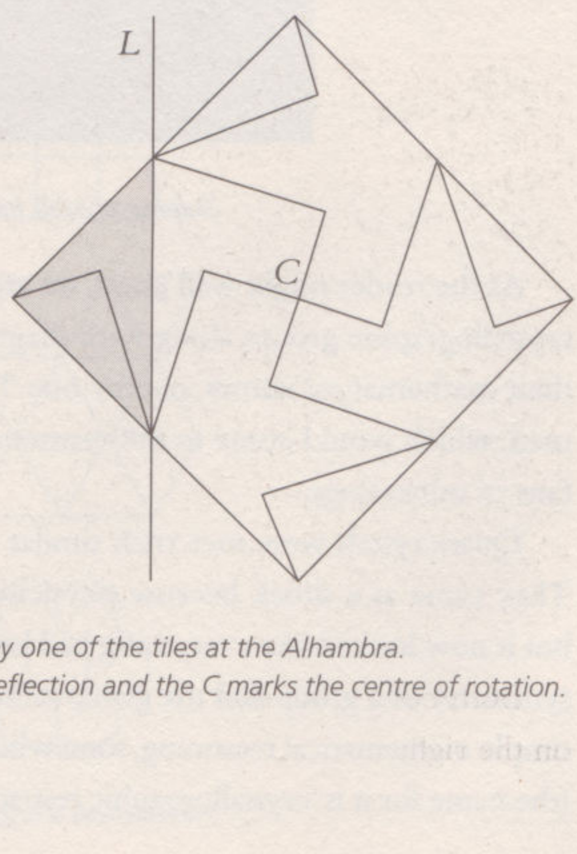
Other examples can be seen in the Alhambra in Granada, Spain, which has become a place of pilgrimage for algebraists. The designs on the friezes, walls, floors, ceilings and so on are all based exclusively on geometric motifs. This means that the motifs are repeated symmetrically in accordance with some of the 17 groups of symmetry planes. Among those in the know, finding the groups of different ornamentations has become almost a sport, and there are several people who claim to have found all 17. However, the debate continues; there are some authorities not convinced by the evidence so far. Finding out if the Alhambra's artisans did in fact put the 17 groups into practice is a rather academic question, as they surely didn't do it inspired by any deliberate calculation or geometry. However, what would, without doubt, be of great interest, would be if they had used 18!





Two examples of mosaic design within Spain's Alhambra.



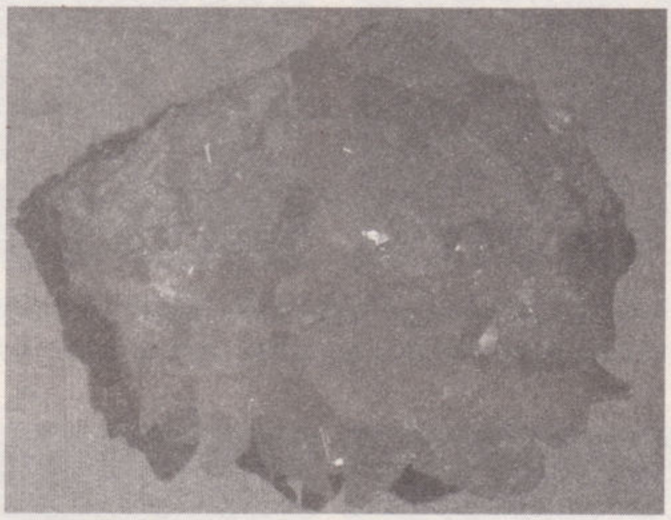


A reproduction of the pattern used by one of the tiles at the Alhambra.
In the figure on the right, the L marks the axis of reflection and the C marks the centre of rotation.

In the 21st century, it is not necessary to create works of art to explore the possibilities of all 17 groups of symmetries. Nowadays software can do it all for you. Just scan in a sketch and tell it which group you want.

Crystals and beyond crystals

In dimension 3, there are 32 possible point groups, but if their possible translations throughout the lattices are taken into consideration, the number shoots up to 230 groups. They are three-dimensional crystallographic groups, also discovered by Fedorov and found in nature. If that figure seems very high, just think of dimension 4, where there are 4,783 crystal habits; in dimension 5 it gets up to millions, and in dimension 6 it reaches 28,934,974...



Sulphur crystals (orthorhombic system).

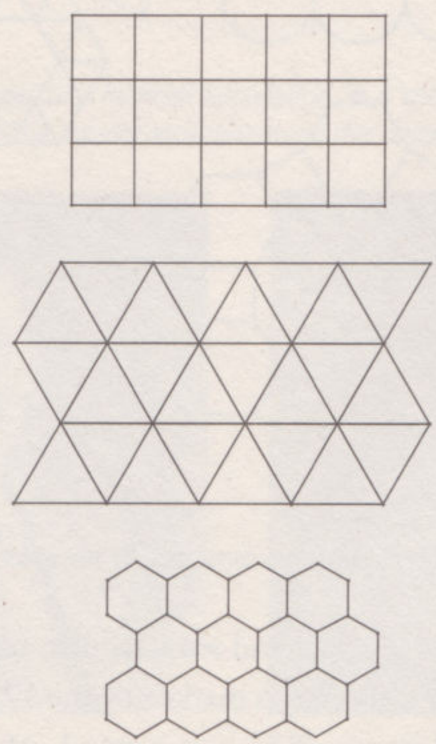
As the reader might well guess, we are not about to provide the list of all the 230 crystallographic groups along with diagrams and photos. Neither shall we give you their mathematical names, one by one. The fact is that three different notations are used, which would come to 690 names and be something for only the most ardent fans of mineralogy.

Quasicrystals were met with similar dismay when they made their appearance. They came as a shock because physicists had been struggling with Fedorov's 230, but it now looked like everything had been put to rest with each crystal obeying the symmetry of a group and the group belonged to one of Bravais' lattices. There was a simple mathematical reasoning, somewhat laborious but at the same time elementary (the name for it is 'crystallographic restriction'), which stated that more lattices could

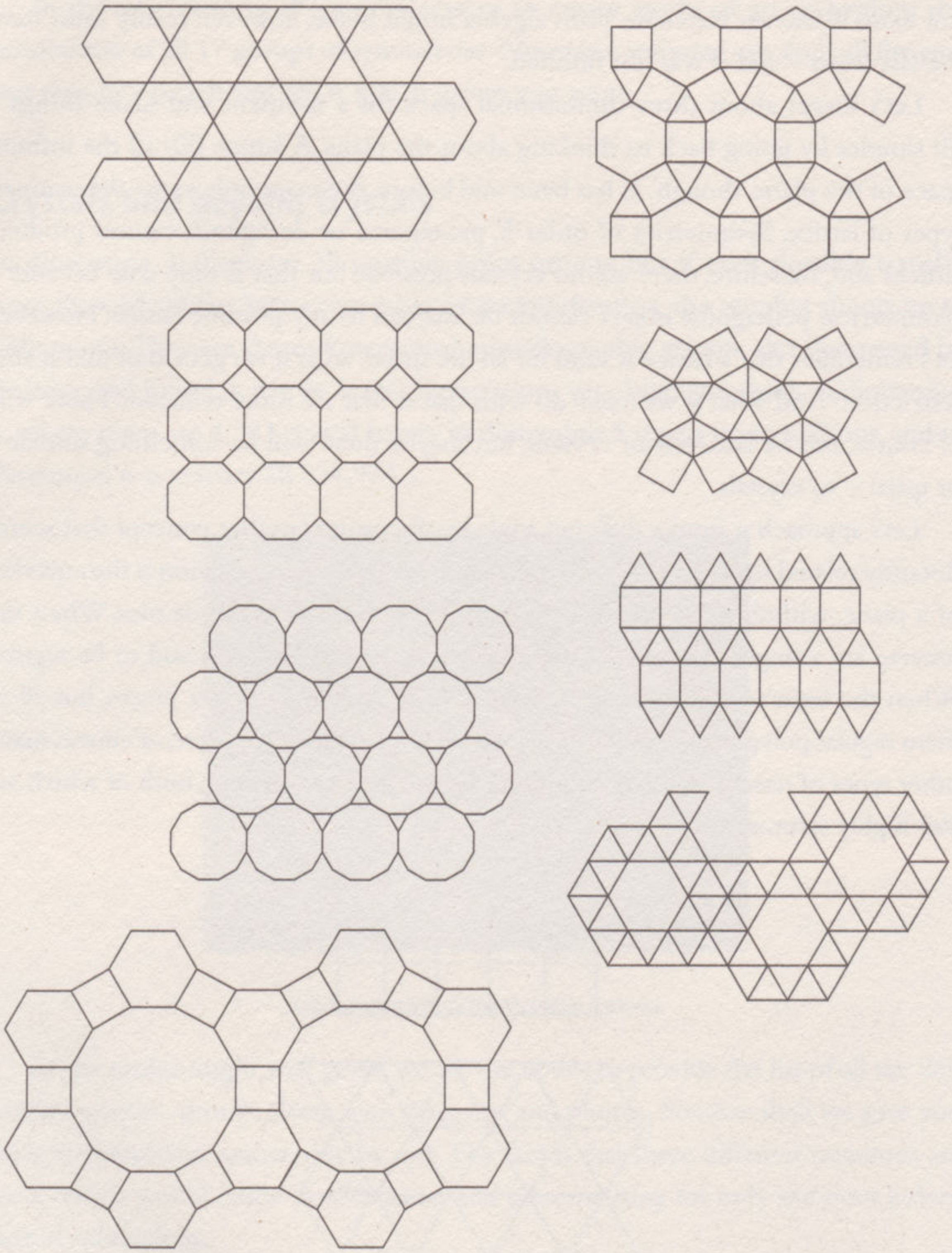
not exist. Whatever elaborate plans algebra might make, however, reality must have the last word – and it was not finished.

Let's forget about three-dimensional space for a moment and make things a bit simpler by going back to thinking about the plane. A lattice fills all the infinite space of the plane, though, as has been said before, there are only some determined types of lattice. Symmetries of order 5, pentagonal or decagonal, cannot produce lattices and, therefore, there are no crystals possible; but that is only true because a symmetrical pentagonal object cannot be adapted to any possible lattice. However, let's remember that a lattice is valid for all the space; what if we get rid of that spatial restriction? And what if we make do with spaces that are more reduced? There will, of course, not be lattices, nor crystals, but maybe there will be something similar – or quasi – to crystals.

Let's approach it from a different angle by discussing another concept that seems distantly related to lattices and crystals, that of tessellation. Tessellation is the covering of a plane, without gaps or overlaps, with pieces called tesserae or tiles. When the tesserae are a single regular congruent polygon, the tessellation is said to be regular. When the tesserae are made up of a number of different, distinct pieces, but all of them regular polygons, the tessellation is called semi-regular. There are, of course, many other types of tessellation that do not belong to these two groups, both of which are also highly symmetrical.



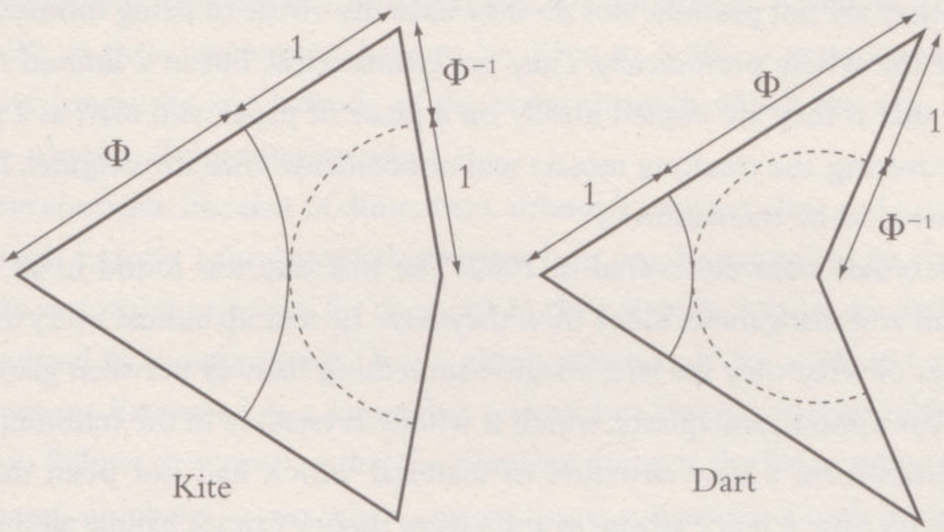
The three possible regular tessellations.



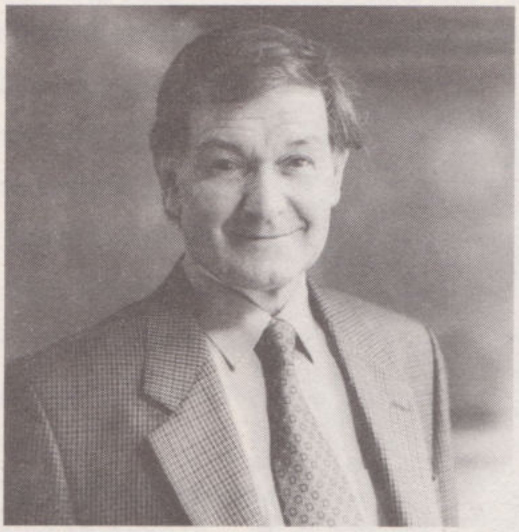
The eight possible semi-regular tessellations.

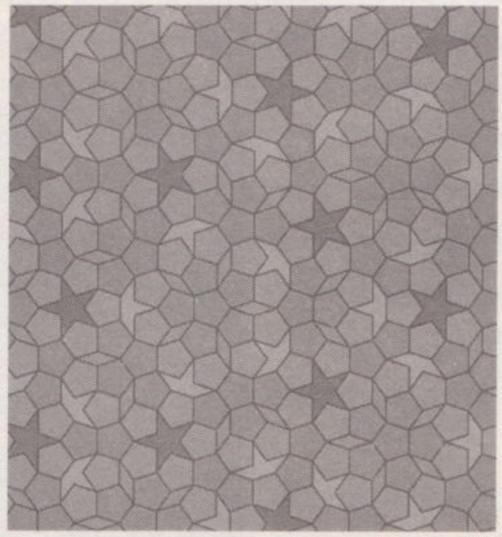
Both types of tessellation, regular and semi-regular, are used to generate periodic tiling, and are subject to the symmetries marked by the 17 plane symmetry groups. In general terms, some tessellations will be symmetrical, others won't, but the periodic tessellations are, of course, always symmetrical.

Let's concentrate on the non-periodic tessellations. An interesting geometric task is to find out the minimum numbers of tesserae necessary to create a non-periodic tiling. The first result came from American mathematician Robert Berger, who in 1966 published an aperiodic tiling of 20,426 different tesserae. Later, as to be expected, the figure started to go down. The philosopher, physicist and mathematician Sir Roger Penrose (b. 1931) surprised the world of maths in 1974. This time it was not for writing something new with his colleague Stephen Hawking – though he has done so before. Instead, Penrose revealed a covering or tiling of the plane that used only two very simple pieces – one concave and the other convex, known as 'dart' and 'kite' – that was not periodic.



Components of the two pieces of a Penrose tessellation. Φ is the golden number $\Phi = \frac{1+\sqrt{5}}{2}$. The interior arcs determine the ratios.



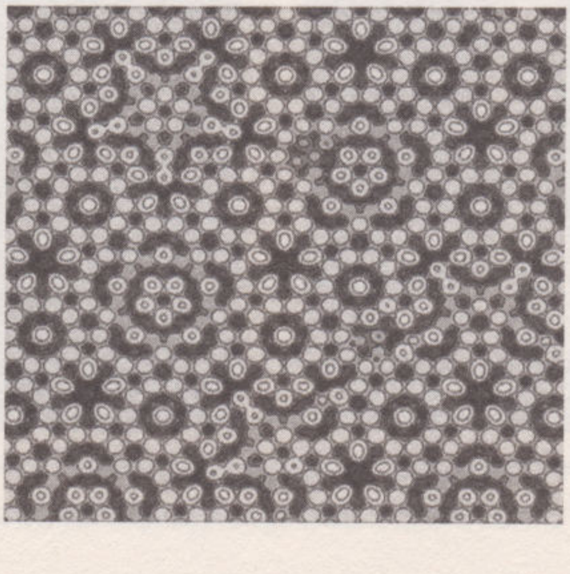


Sir Roger Penrose, pictured in 2007, and an example of the type of tiling he devised.

In other words, the plane could be covered with only two pieces but the resulting mosaic was not periodic. That meant it could not be infinitely repeated, and it was not symmetrical across the whole plane. Taking that as a starting point, Dov Levine and his boss, the cosmologist Paul Steinhardt, found a tiling formed by rhomboidal cells that was not planar but spatial – and also non-periodic. It did not fit in with any lattice, however, and therefore such a covering was not periodic, nor did it symmetrically cover *all* of space like Federov's 230 groups could.

Coverings of the Penrose type are not periodic and do not reproduce the same scene *everywhere*. They lack a symmetry that is predictable at any point, but they are symmetrical in very extensive areas; in actual fact, in any area, however big it is. But, crucially, they are not periodic nor do they have the virtue of being infinite, and do not cover the whole periodically. They are symmetrical, but in a limited form, in the sense that if they are copied ideally on a piece of paper, and used as a plan for another covering, the resulting mosaic will not coincide with the original. They are not symmetrical by translation.

Quasicrystals were discovered in 1982. The first one was found in an alloy of aluminium and manganese. Since then they have turned up almost everywhere. To get an idea of what they are like, imagine something halfway between glass, which is totally amorphous, and quartz, which is wholly crystalline in the traditional sense. Quasicrystals form a new structure of material which had not been described before. That's how science advances and group theory cannot handle all the secrets of symmetry. It will be necessary to go beyond groups to respond to these new challenges and quantum physics may well make such contributions in the future. It shows how important groups and symmetry are, that even when their laws are not kept (known as a 'breach') it means a new challenge appears.



An atomic model of the quasicrystal formed by an alloy of silver and aluminium.

Atoms and groups

There are finite groups, like almost all those we researched in the first chapters. The polygon dihedral groups and the polyhedron groups, and so on, but there are also infinite groups, like the two-dimensional plane itself, which has infinite isometries that leave it invariant.

In fact it is not necessary to go so far. Let's take a circle, some would say an insignificant geometric figure albeit one that has infinite symmetries. It has so many symmetries, including simple rotations, that it is not enough to say that they are infinite. We need to say, so as to get a rough idea, that the group of rotations of a circle has dimension 1, as 1 is the number of parameters necessary to define any movement. If we had chosen a spherical ball, the dimension would be 3 (take note it's not 2!), as three parameters have to be given to define a gyratory movement. (They are longitude and latitude of the point through which the axis of central rotation passes and the rotation angle.)

These concepts, like that of dimension, although they are clear and necessary, are not enough to solve a fundamental question: how are the groups to be classified? Is there any reasonable criterion for doing it? In the previous chapter, we defined what is understood by 'simple group'. It is a group which only has itself and its identity as a subgroup. Expressed in a somewhat inexact but very comprehensible way, the definition follows the criterion that the simpler a group is, the fewer subgroups it has; and extreme simplicity occurs when a group has no subgroups. Simple groups are the least complex groups possible, the basic and only ones that cannot be broken down.

Simple groups are like the elements in the universe of groups. They are indivisible. A simple group G does not have normal subgroups N to make it possible to "divide" G. There is therefore no quotient group G/N. Such as quotient in impossible.

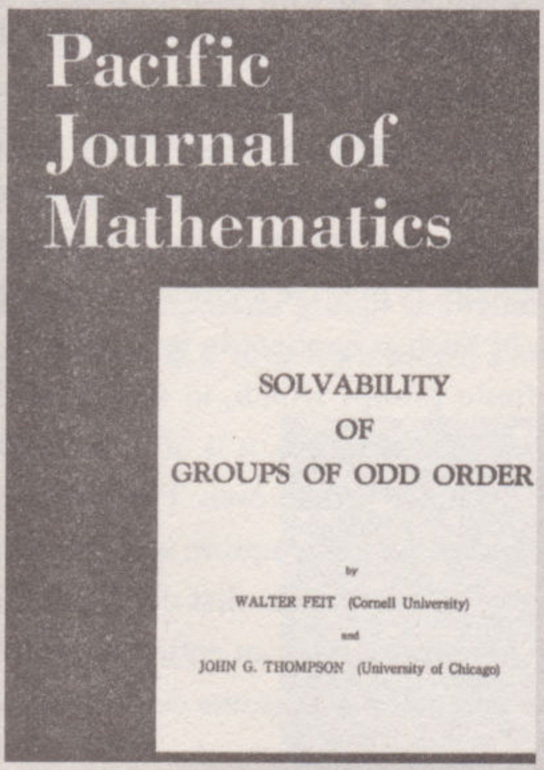
For the moment, let's deal with the finite groups which, in theory, is the easiest task. When a group is finite and cyclic, if it is simple, it is of prime order. If the simple group is finite but not cyclic, its order is then even. This statement is a beautiful result known as the theorem of Feit and Thompson, who proved it in 1963. It's curious that the formulation is very brief, but its first demonstration took up no fewer than 254 pages! It was so long that it filled an entire issue of the *Pacific Journal of Mathematics*. It was a pleasant omen of the awesome demonstrations to come.

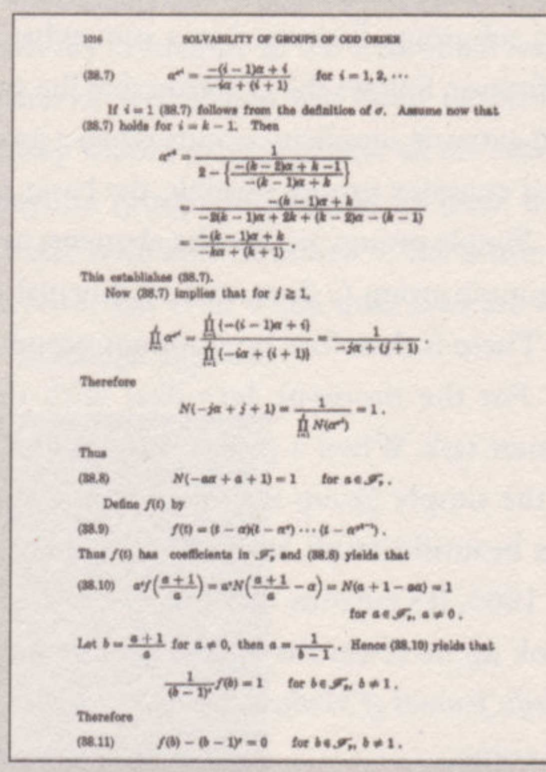
The following years have seen more development in the classification of the simple groups, whether finite or not, though here – as we have already said – we shall limit ourselves to the finite ones. It is not that classifying the infinite groups is an impossible

JOHN GRIGGS THOMPSON (b. 1932)

This American mathematician has always worked in the field of algebra and, in particular, on group theory. His work in this field has been so outstanding that he was awarded the Fields Medal in 1970 and, as we saw in Chapter 3, the Abel Prize in 2008. He and Jean-Pierre Serre (b. 1926) are, for the moment, the only mathematicians who hold both awards. (Remember that the Fields Medal is given to the best researcher under the age of 40, and the Abel Prize is to honour the best mathematicians in the world based on their whole career.)

Thompson's best-known result is the theorem that joins his name with that of his colleague Walter Feit (1930–2004) – the Feit-Thompson theorem –which was the first significant step to classifying simple groups. Thompson also discovered the sporadic group that bears his name, i.e. the group known as *Th* (o $F_{3|3}$ o F_3) of order $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31 = 90{,}745{,}943{,}887{,}872{,}000 \approx 9 \cdot 10^{16}$. One result of his, which specialists will appreciate in its entirety, states that this sporadic group – of an almost unimaginable size – is a Galois group of some integer equation. He does not say which, but such an equation must be colossal.





1016 SOLVABILITY OF GROUPS OF ODD ORDER

(38.7) $$\alpha^{\sigma^i} = \frac{-(i-1)\alpha + i}{-i\alpha + (i+1)} \quad \text{for } i = 1, 2, \cdots$$

If $i = 1$ (38.7) follows from the definition of σ. Assume now that (38.7) holds for $i = k - 1$. Then

$$\begin{aligned}\alpha^{\sigma^k} &= \frac{1}{2 - \left\{\frac{-(k-2)\alpha + k - 1}{-(k-1)\alpha + k}\right\}} \\ &= \frac{-(k-1)\alpha + k}{-2(k-1)\alpha + 2k + (k-2)\alpha - (k-1)} \\ &= \frac{-(k-1)\alpha + k}{-k\alpha + (k+1)}.\end{aligned}$$

This establishes (38.7).

Now (38.7) implies that for $j \geq 1$,

$$\prod_{i=1}^{j} \alpha^{\sigma^i} = \frac{\prod_{i=1}^{j}(-(i-1)\alpha + i)}{\prod_{i=1}^{j}(-i\alpha + (i+1))} = \frac{1}{-j\alpha + (j+1)}.$$

Therefore

$$N(-j\alpha + j + 1) = \frac{1}{\prod_{i=1}^{j} N(\alpha^{\sigma^i})} = 1.$$

Thus

(38.8) $$N(-a\alpha + a + 1) = 1 \quad \text{for } a \in \mathscr{F}_p.$$

Define $f(t)$ by

(38.9) $$f(t) = (t - \alpha)(t - \alpha^{\sigma}) \cdots (t - \alpha^{\sigma^{q-1}}).$$

Thus $f(t)$ has coefficients in $\mathscr{F}_p$ and (38.8) yields that

(38.10) $$a^q f\left(\frac{a+1}{a}\right) = a^q N\left(\frac{a+1}{a} - \alpha\right) = N(a + 1 - a\alpha) = 1 \quad \text{for } a \in \mathscr{F}_p,\ a \neq 0.$$

Let $b = \frac{a+1}{a}$ for $a \neq 0$, then $a = \frac{1}{b-1}$. Hence (38.10) yields that

$$\frac{1}{(b-1)^q} f(b) = 1 \quad \text{for } b \in \mathscr{F}_p,\ b \neq 1.$$

Therefore

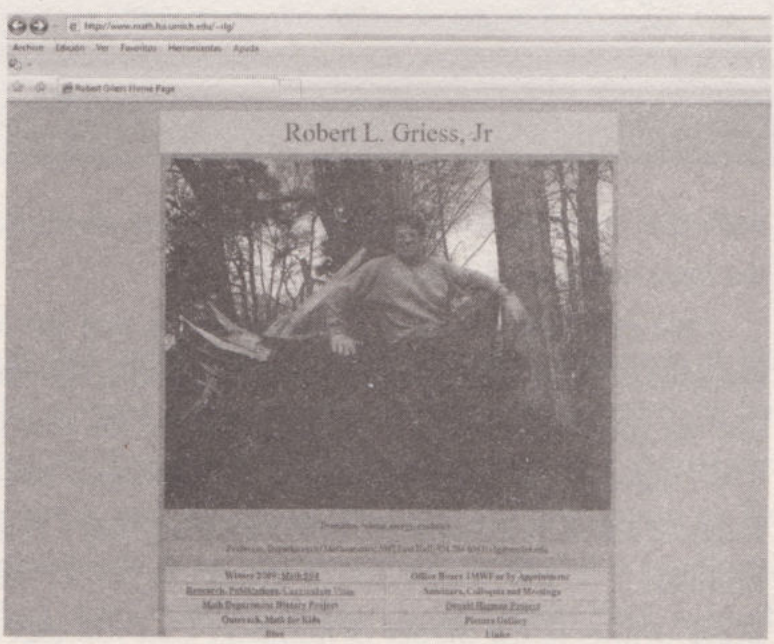
(38.11) $$f(b) - (b-1)^q = 0 \quad \text{for } b \in \mathscr{F}_p,\ b \neq 1.$$

The front cover of the Pacific Journal of Mathematics *and one of the pages of the extensive article written by Feit and Thompson in 1963.*

task - far from it - but it requires long, labyrinthine explanations on Lie groups, which are possibly too specialised. As for the finite groups, they can be:

- Cyclic groups, type $\mathbb{Z}_p$ with p prime.
- Alternating groups for $n \geq 5$, already covered in previous chapters and denoted by A_n.
- Simple Lie groups, which we will cover in their corresponding section.
- Sporadic groups, of which there are 26; they do not follow any pattern and while, at first, they were described as oddities, now they are regarded with respect, as ever more interesting things are emerging from deep inside them.

The sporadic groups range from the Mathieu group, the lower order one, as it 'only' has $7{,}920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ elements, to the group known as the *Monster* which has $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 =$ 808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000 elements. (In case you were wondering that is more than the total number of atoms in the universe.) It was built by Robert Griess (born 1945) in 1980, and its author calls it "the friendly giant", a name that became widespread amongst specialists. It is often said – perhaps in a good-hearted attempt to simplify things – that it corresponds to the group of certain rotations in the space of dimension 196,883.



The homepage of mathematician Robert Griess, the builder of the Monster.

In 1990, the uniqueness of the *Monster* was proved. There is only one, thank goodness. This enormous group is denoted in specialised publications as M. Of the 26 sporadic groups, 20 are subgroups (note, they are non-normal!) of M; specialists call the six remaining ones, which are outside M, 'pariah groups'. The next sporadic group, the one that comes after M in size, is dubbed *Baby Monster (B)*, with a mere 4,154,781,481,226,426,191,177,580,544,000,000 elements – insignificant compared to M.

Sporadic groups are full of surprises and a minefield to be traversed with care. Take, for instance, the peaceful *Monster:*

One day, Conway and Norton posed a conjecture, the result of contemplating the irreducible dimensions of M and of the coefficients of a development of Fourier on the function called $j(\tau)$. The two mathematicians' perplexity is understandable faced with a coincidence, which is not evident at first glance, between two things that seemed not to have anything to do with each other, some coefficients of one series and the dimension of certain representations of M. They noted that

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \ldots$$

(where $q = e^{2\pi i\tau}$ has been used to abbreviate) and that, meanwhile, the dimensions fulfilled

$$\begin{aligned} 1 &= 1 \\ 196884 &= 196883 + 1 \\ 21493760 &= 21296876 + 196883 + 1 \\ 864299970 &= 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1 \\ &\vdots \end{aligned}$$

with the numbers on the right being linear combinations of the dimensions that we have mentioned before.

One probably has to be Conway to see that the above is too beautiful to be a coincidence and consequently to issue a conjecture. But that is what he did. This is not the place to reproduce it, but suffice to say that it even involves string theory. The conjecture, humorously called *The Monstrous Moonshine*, was proven in 1992 by Richard Ewen Borcherds (b. 1959) winning him a Fields Medal. One last point: The numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59 and 71 are the fifteen called super-primes in the theory of numbers. Can you spot any link with the order of M?

JOHN HORTON CONWAY (b. 1937)

Conway is one of the most popular, multitalented, intelligent and out-of-the-ordinary contemporary mathematicians. He is a prolific author of books and creator of videos which range from intellectual algebra texts and philosophical speculations to recreational mathematics, a field in which he is something of a star. He was born in England and holds the John von Neumann professorship at Princeton, USA.

It is difficult to say which of Conway's creations will stand the test of time, but there will be a considerable number, and in many different fields. He has been responsible for finding four groups while working independently, and three more working with colleagues. He has also written an atlas of the finite groups. He has worked on quantum physics and the Doomsday algorithm, a kind of universal calendar. In automata theory, he invented the very popular *Game of Life*; in number theory, surreal numbers, and in quantum theory, he was responsible for the surprising "Free Will Theorem", dedicated to exploring the wonders of the quantum world. He has done important work on quaternions, knot theory and the combinatorial theory of polyhedra; proved that every integer is the sum of at most 37 fifth powers... All in all, he is a scientist who defies all categorisation.

The Honourable Professor John "Horned" Conway, sporting an Alexander "horned" sphere, a geometrically infinite surface. It was Conway who invented the nickname Monster *to refer to the M group.*

Lie groups

The Norwegian Sophus Lie (1842–1899) did not suffer the same privations as his struggling compatriot Abel; he worked as a university professor in Germany. In his work on differential equations, he developed some special groups of a geometric nature which today bear his name and which, at the time, did not appear to be destined for the limited, yet worldwide, fame that they enjoy today. Other intellects, such as those of Wilhelm Killing (1847–1923) and Élie Cartan (1869–1951) carried

on with his work and Lie groups are today a fundamental instrument not only in algebra, but also in physics and, in particular, quantum physics. The 2008 Abel Prize winner, Jacques Tits (b. 1930), is a specialist on this topic.

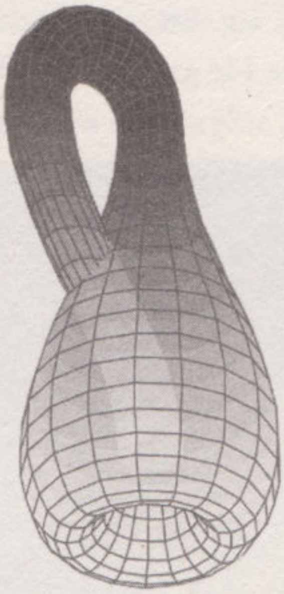
Sophus Lie the Norwegian developer of Lie groups.

A Lie group is a differentiable variety (to imagine one, think of a smooth surface, with no folds, or of a sphere, a helix or a doughnut) the elements of which constitute a group with operations that are also differentiable. In actual fact, at first sight, it is not at all strange. However, what happens is that under this innocent umbrella, hide unsuspected, very uncommon, and even finite, differentiable varieties which, on account of their complexity, will not be dealt with here. Lie groups are the stuff of higher algebra and we shall not put the reader through a description and study of them here; anyone who wants to, can take an in-depth look at them in the Appendix.

The Erlangen program

Felix Klein (1849–1925) was not only recognised in his lifetime as a top mathematician, but was also known as a workaholic, a fanatic in his work who went without sleep and even took drugs to stay awake and achieve more results. So great was his enthusiasm that his body finally decided it had had enough and, in 1882, he suffered a breakdown. He never really recovered.

Nevertheless, we owe many advances in mathematics to Felix Klein, such as the incorporation of derivatives and integrals to secondary education, i.e. to non-university education, plus, in the strictly mathematical sense, many important concepts and ideas. He also created a topological structure, almost like a toy, called the Klein bottle.

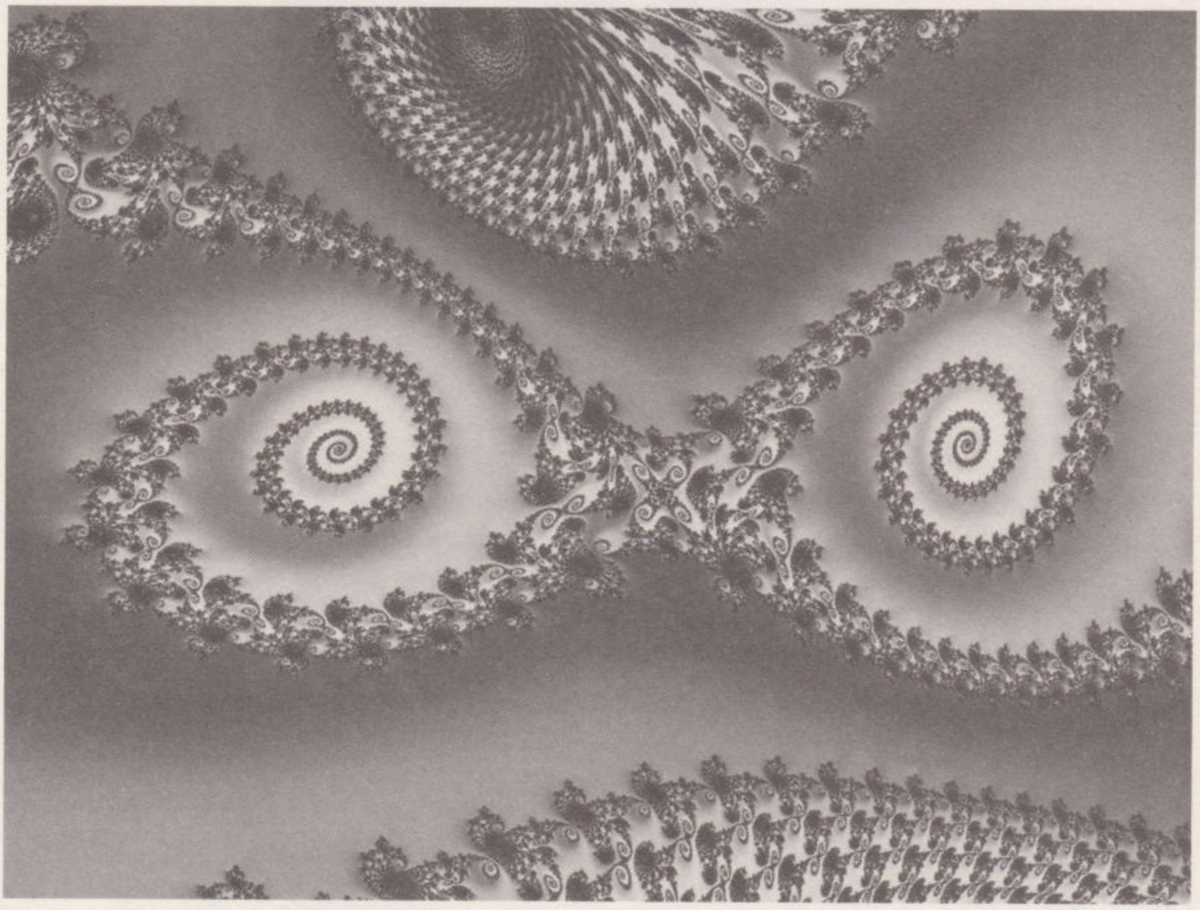


A Klein bottle is a vessel that contains no volume whatsoever. It has no interior or exterior. In dimension 4 it does not penetrate itself as seen here. The term bottle *is in some ways an apt, but not wholly adequate, term. It comes from a mistranslation of the German term* Kleinsche Fläche. *While* Flasche *means 'bottle',* Fläche *means 'surface'.*

When mathematicians speak of Klein, they also immediately think of Erlangen's program, a new approach to geometry that Klein presented in one of his speeches at the University of Erlangen, where he first became a professor. Though today it is so completely accepted that the whole scientific world takes it for granted, Klein's viewpoint was, at the time, a complete novelty. What he intended – and achieved – was to synthesise in one single idea all the diverse meanings of the word 'geometry'. In Klein's day, there were many types of geometries: Euclidean, non-Euclidean, metric, projective or affine geometries, and others. The central idea behind the

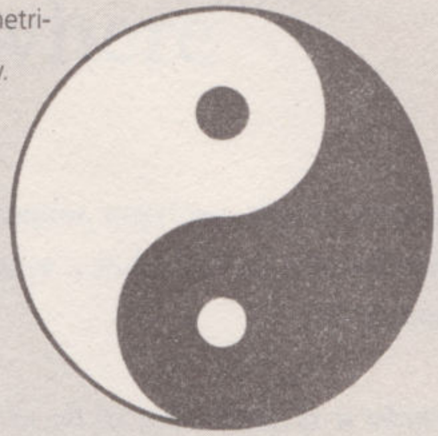
Erlangen program is to concentrate on a group G and on all its symmetries, on all the transformations of G which leave some basic properties invariant. That is what forms the contents of the geometry determined by the group.

If the group chosen is that of the Euclidean isometries, we get Euclidean geometry. If we were to take the metric geometry of Lorentz-Einstein and the group chosen were the Lorentz transformation group in dimension 4 (Minkowski space), the result would be geometry of the theory of relativity. If the group we choose is the affine transformation group (which make vector $\boldsymbol{av} + \boldsymbol{b}$ from another vector $\boldsymbol{v}$), we would get an affine geometry. This a geometry in which the similarities, as well as the rotations and reflections, would be invariant and in which spiral objects, such as nautiloids and their shells, would be considered symmetrical, even though they are not considered symmetrical according to the conventional symmetries of the Lie group $SO_2(\mathbb{R})$.



The world of fractals, which gathers together self-similar objects, is an ideal world in which to study the affine group up close.

AN ANTI-SYMMETRICAL SYMMETRY



The symbol for the concept of of yin and yang is not symmetrical, but it has a certain something that suggests symmetry. It is so asymmetric that it really deserves to be symmetrical. In fact, yin becomes yang (and vice-versa) with one turn, followed by a change of colour. In mathematics, a function f that verifies that

$$f(x,y) = -f(y,x) \text{ for every } x \neq y$$

is called anti-symmetric.

From a practical point of view, two isomorphic geometries, however hidden that isomorphism may be, are basically the same geometry. For instance, the geometry of the complex projective line is the same as that of the real hyperbolic plane. Their groups are indistinguishable except for isomorphism. It is enough, then, to study just one geometry, not two.

Chapter 6

Symmetry Everywhere

Theorems have to be genuine, surprising, elegant, intriguing, rigorous, creative... and, above all, comprehensible.
H. Zeeman

The world of physics is where symmetry shines in all its splendour. It is obvious every day in mirrors and optical phenomena and is found in familiar items such as drill bits or the kaleidoscope. However, it is present in more hidden areas, in more complex theories, particularly in the quantum field, that infinitely small, but real, world so distant from our daily experience. The quantum scale is also beyond our perception, which makes it enormously difficult to understand. In fact, some concepts in physics reach such dizzy heights in scientific thinking that they are regularly greeted with disbelief. We'll look at the subject from several points of view to help you not only believe, but also understand.

In physics, symmetry occupies a special place, some would say a privileged one. Any situation – the result of an observation or consequence of a principle – in which a mathematical entity is invariant after a change, denotes symmetry.

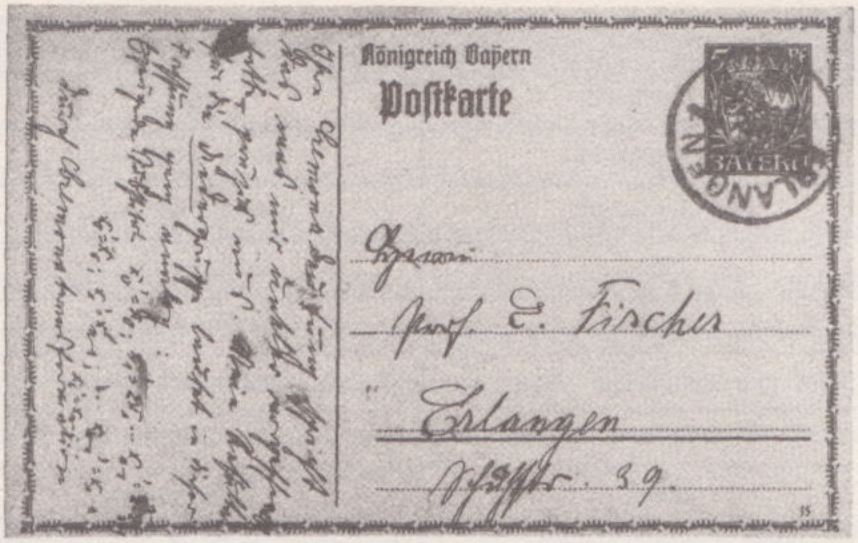
Symmetry, symmetry, you have a woman's name

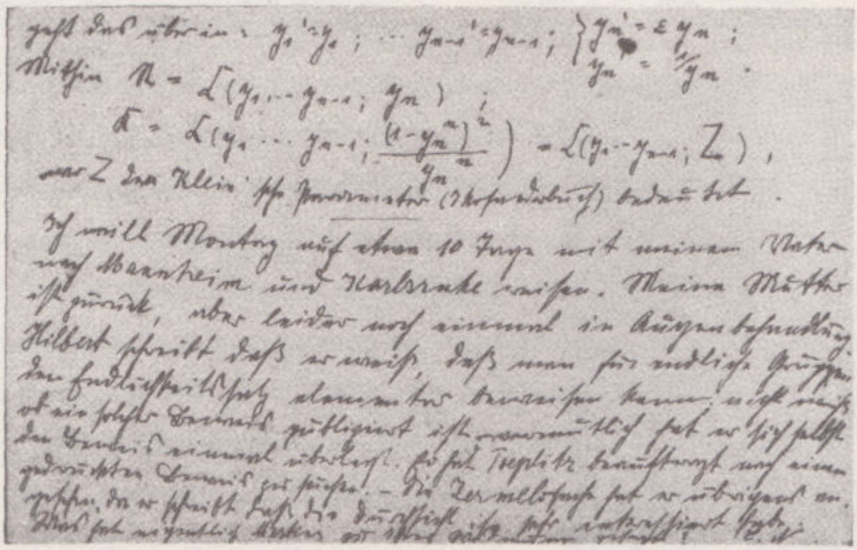
Amalie Emmy Noether (1882–1935) had to struggle all her life against the prejudices of the time. A woman, however intelligent she might be, was not accepted into the scientific world; her place was in the home looking after the children. David Hilbert (1862–1943), the indisputable leader in *fin de siècle* mathematics, spent much of his time and put his professional prestige on the line, to ensure that Noether, the daughter, incidentally, of a distinguished colleague, Max Noether, should be given a professorship at the University of Göttingen. She moved there in 1915, but it was not until 1923 that she was given a non-official position as associate professor. And as for the problems she had attempting to get paid for her work, perhaps the less said the better, as it belongs to the realm of the incredible. She was given no recognition by her colleagues, and despite her indisputable status as the top female figure in

the world of mathematics, Noether was never able to ascend to her rightful place at the top of her profession. In 1930, a rather ridiculous situation arose when her pupil Hermann Weyl was appointed professor while she remained a simple teacher. Despite this grotesque abuse, the establishment did nothing.

Albert Einstein and many other scientists considered Amalie Emmy Noether to be the most important woman in the history of mathematics. She proved the fundamental theorem of invariance and created the modern theory of ideals, while researching ever deeper into the mysteries of algebra.

When Nazi tyranny forced Jewish scientists out of the country, Noether emigrated to the United States. Even there, in the Land of the Free, she had to make do with a position at Bryn Mawr College, an educational institution for girls, despite the fact that she deserved a place at Princeton, alongside Einstein, Gödel, Von Neumann and the other European emigres. Emmy did give lectures alongside them there, but always as an invited guest, not as a member of the Institute for Advanced Study.





Emmy Noether even used postcards to set down her reflections on mathematics, as can be seen in this message posted in 1915 to her friend, Professor Ernst Fischer.

Noether's most significant contribution to symmetry is her somewhat startling invariance theorem. This theorem says that, in physics, every symmetrical mathematical formula is equivalent to the existence of an invariant physical entity. And vice versa.

You guessed it, the algebraic demonstration of this theorem is difficult, long, and complex. But it is astounding, both in its reasoning and its result. It has been described as "one of the most important theorems ever proven" and deserves all the praise it receives. As an example, overleaf is a short piece formed by the pairing invariant-symmetry/conservation resulting from Noether's theorem:

Invariant	Preserved physical entity
Time translation	Energy
Space translation	Linear momentum
Spatial rotation	Angular momentum
CPT	Product of parity
$U(1)$	Electrical charge
$U(2)$	Electroweak force
$SU(2)$	Isospin
$SU(3)$	Quark colour
$U(1) \times SU(2) \times SU(3)$	Standard Model of quantum physics

The symmetry denoted by the initials *CPT* refers, with a certain approximation, to the structure of the universe we live in. The charge of the particles of which it is formed can be positive or negative, and that configures the charge conjugation *C*; the space may be the current one or its inverse image in an imaginary mirror, and that defines the inversion of coordinates or parity transformation *P*; and time can move forwards or backwards, and that is referred to by the temporal inversion *T*. Each term may be to the value $+1$ or -1, depending on the parity, and what the theorem specifies is that the product of the parities is preserved.

Symmetry in quantum theory

The Lie groups that appear at the end of the brief table above are 'gauge' symmetries, which is a common term in university texts but which needs to be used with ease in the world of higher physics. In 1918, Hermann Weyl started to study a physical symmetry, which he called 'gauge' symmetry or measurement symmetry, and which linked, with the aid of group theory, electromagnetism to gravity. Although his attempt to consider the two fields as geometric properties of space-time was no more than a good attempt (and a failed one), the concept had now been born.

The most accurate way to define gauge symmetry is the purely mathematical one. Imagine a physical theory that is based on the use of fields, like the electromagnetic or gravitational, and let's suppose that there is a series of transformations that form a group *G*, the group that we shall call gauge. The theory is a gauge theory with gauge *G* group when the Lagrangian field is an invariant of *G*.

That is a very precise definition, but incomprehensible to the uninitiated. To make it easier to understand, a previous step is necessary. We can define certain elementary particles, called fermions, named after the Nobel Prize winner Enrico Fermi (1901-1954), as being characterised by having a half spin.

Behind that brief definition the fermions are divided further into the quarks and leptons. There are twelve of them if their anti-particles are counted, and six if they aren't. They answer to the diverting names of up quark, down quark, charm quark, strange quark, top quark and bottom quark. There are six leptons (plus six anti-particles) and though their names may not be particularly descriptive, they

HERMANN WEYL (1885–1955)

Hermann Weyl was one of the great intellects of the 20th century. His legacy was not limited to mathematics or physics; he was also a magnificent writer, although his works were more philosophical than popular. He had the greatest impact by tirelessly formulating many new concepts that opened up new channels of research.

Born in Hamburg, Germany, Weyl went on to study at the University of Göttingen along with luminaries of physics and maths such as Hilbert and Minkowski. With his authoritative mastery of almost all fields of mathematics, he has been described as the "universal scholar". Offered the opportunity to work with Einstein and Schrödinger, he was also an authority on the theory of relativity. Later, as was the case with so many other German scientists, pressure from the Nazis prompted him to emigrate to the United States, where he entered into the prestigious Institute of Advanced Study at Princeton.

He studied Reimann varieties, invented *gauge* symmetry – though his physical interpretation of it turned out to be incorrect – and harmonised the basics of quantum theory with classical physics, with groups playing an ever-present role. He also made important incursions into other mathematical fields, such as harmonic analysis.

include the electron, the positron, the muon, the tauon, and the neutrino. Let's imagine a quantum theory of fermions that explains the interactions among them as local transformations of a group G. The group is called gauge, and the theory is gauge theory. Summing it all up in a few words, we could say when the term gauge is used, there is a symmetry group among particles. The mathematics behind it all is differential geometry.

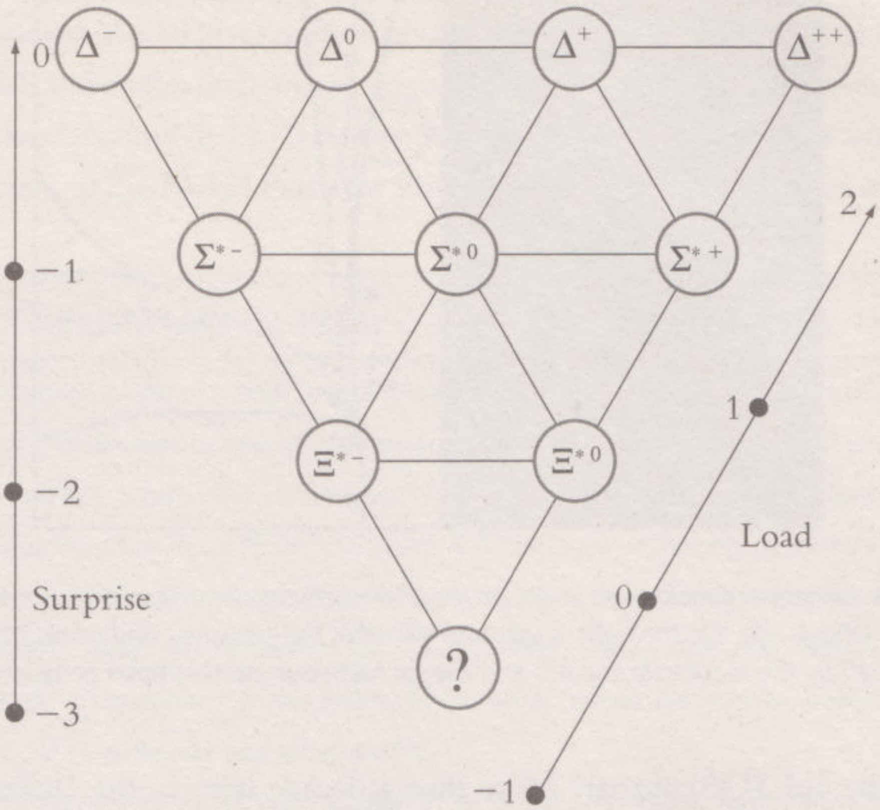
Another, perhaps less aggressive way to approach the concept of gauge is to go back to the origins: a field theory is of gauge symmetry when different configurations of the same field – induced by the action on the field's mathematics by a symmetry group (which we shall call gauge) – give invariant observable measurements. In that way, both the word 'invariance' and the word 'measurement' are made to stand out.

Gauge symmetry is complexity incarnate: It involves complex numbers (in quantum physics they are king), partial derivatives and curvilinear integrals, some algebraic objects called Lie algebras (linked to Lie groups), matrix calculation (including infinite matrices), and certain entities known as commutators which lead to inequalities of the kind

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

for a particle of average position x and average momentum p, where h has a universal numerical value, called the Planck action quantum. This formula bears the hallowed name of "Heisenberg's Uncertainty Principle", which says that it is *impossible* to know, *at the same time*, how a particle is moving and where it is.

The comprehension of gauge theories within quantum physics is not straightforward, but unless someone should demonstrate that they are false (never say never), they have been proven to be right so often that they are unlikely to be replaced by other theories. This was dramatically highlighted by the now well-known story of the Nobel Prize winner Murray Gell-Mann (b. 1929) and particle Ω^-. In the 1960s, Gell-Mann and the Israeli politician and military leader Yuval Ne'eman (1925-2006) were involved in research into hadrons (a type of subatomic particle). They observed that many particles could be arranged into very elegant groups if attention was paid to various quantum quantities, like strangeness, charge and spin, and in such a way beautiful structures could be created, such as the figure opposite:

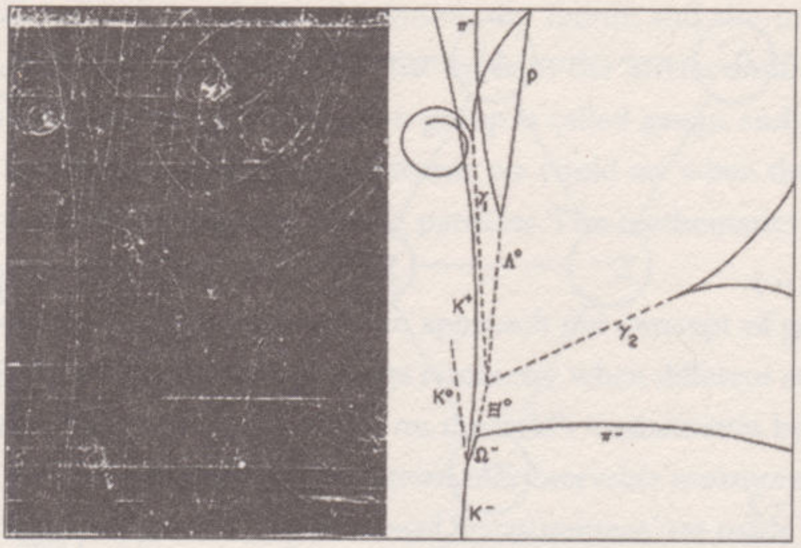


If the particles are placed on a pair of axes of coordinates (the ordinates and abscissa indicate the magnitudes being considered in the diagram), Gell-Mann thought that there was a gap at the bottom. Something was missing – the so-called omega minus particle (Ω^-). The search began and the Brookhaven lab found it 1964.

It is obvious (now, that is, but not then) that there is a gap. Gell-Mann and Nee'man postulated the existence of a particle that would fill it, rather like Mendeleev did a century earlier with the elements in his periodic table. The Lie group that supported this *gauge* theory was $SU(3)$, and in 1964 the particle Ω^- was indeed discovered. And so everything fitted with $SU(3)$.

SPIN

One of the simplest models of atoms describes a central nucleus with orbiting electrons, converting atoms into miniature solar systems. Within that context, the allegorical concept of spin was created to describe the rotation movement – or rather, the angular momentum – of some particles as they turned over and over. Present-day quantum mechanics considers spin as just another quantum number. Elementary particles that have full spin are called bosons. Those with half spin are called fermions.



A bubble camera at Brookhaven in Rochester, USA, captures the first evidence of the elusive particle Ω^- (above left). On the right, a diagram identifies the trajectory and traces of the particles left by the successive impacts and the spontaneous disintegration produced.

But why did $SU(3)$ appear? More than a decade later, in the 1980s, the riddle was considered solved by accepting that each hadron (the protons, neutrons etc) was formed by three smaller particles – three of the quarks in fact.

Supersymmetry

There are four forces governing nature, and all of them are perfectly mathematised in vectorial form as field theories. By greatly simplifying, we can reduce the descriptive fields of what surrounds us to four: the gravitational field, electromagnetism, the weak interaction and the strong interaction.

The first two forces do not need any explanation as they are already familiar. The other two are always present within us as they are the forces that act on our atoms. The so-called weak force is due to the interchange of bosons (elementary particles without spin) W and Z – they have a magnitude of order 10^{-11}. As the other force, the one that holds protons and neutrons together, is some 10^5 times stronger, it is called the strong force. It definitely is really strong, equal to some 100 times the electromagnetic force and 10^{39} times the gravitational – albeit acting over a minute distance by comparison. For sure, our atoms are well and truly secure, and it takes a lot to separate their components. That is the energy released in atom bombs, but that's another story.

The Standard Model gives quite a satisfactory joint explanation of the three latter forces or fields and unifies them in one single theory. Its definitive formulation was drawn up by Sheldon Glashow (b. 1932), Steven Weinberg (b. 1933) and Abdus Salam (1926-1996), winners of the Nobel Prize for Physics in 1979. The theory that managed to

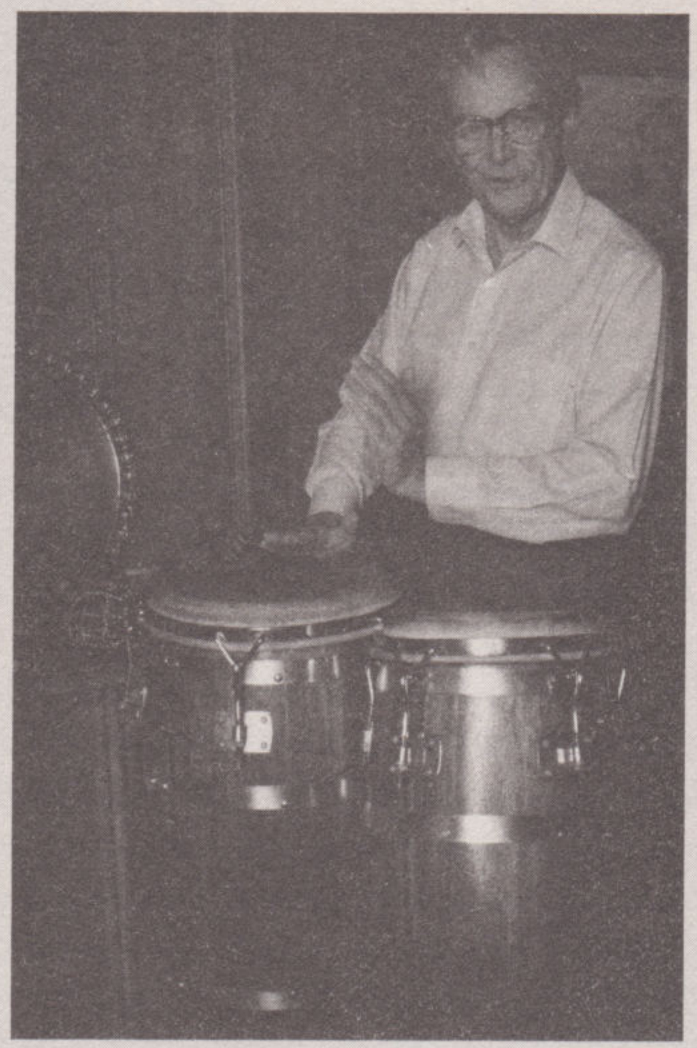
MURRAY GELL-MANN (b. 1929)

A North American physicist – and, what's more, child prodigy – Murray Gell-Mann made his best-known contribution to science by introducing the quark into the world of particles. In 1969, his work in this field earned him a Nobel Prize. Gell-Mann took the word 'quark' from a piece of the James Joyce novel *Finnegan's Wake*. But that was not the only term he coined. Not long before, he had noticed that subatomic particles could be arranged into groups of eight (and of ten) and named this mechanism the *Eightfold Way*, making an analogy with the "Noble Eightfold Path" of Buddhism. It was explained later that behind the mystical-physics play on words were *SU*(3), symmetry and group theory.

Although the above would have been enough to confirm his reputation, Gell-Mann made many more contributions to his field. One of his first incursions into the universe of elementary particles resulted in the discovery of a new quantum 'number', strangeness. In 1972 he published – in collaboration with Harald Fritzsch (b. 1943) and Heinrich Leutwyler (b. 1938) – the still-valid theory of quantum chromodynamics. He also worked with another leader in of quantum physics, Richard Feynman, on the vectorial structure of the weak interaction. One of his works is *The Quark and the Jaguar*, a general audience book on complexity.

Murray Gell-Mann (near left) receiving his Nobel Prize from King Gustaf VI on December 10, 1969.

RICHARD FEYNMAN (1918–1988)



Richard Feynman was an unusual scholar. Few physicists, and even fewer of the very important ones, get the chance to be a TV star, and even fewer become famous for playing the bongos, but Feynman achieved both. His life reads like a novel and has been set out in a number of biographies. Feynman was a great communicator, a skilful teacher and a maverick scientist, who asked all kinds of questions, at times very awkward and always very original.

Feynman's scientific career began at the legendary headquarters of the Manhattan Project at Los Alamos. A good indication of his character is that he spent his leisure time breaking the combination codes of locked desks and cabinets and leaving notes inside.

Feynman became deeply involved in quantum physics. He was awarded the Nobel Prize in 1965 for his studies on renormalisation and quantum chromodynamics. He invented Feynman diagrams, auxiliary graphs of invaluable help in research on particle physics. He also studied superconductivity, weak interaction – with Murray Gell-Mann – quantum computers and nanotechnology. Perhaps his greatest moment in the spotlight came when he was a member of the inquiry commission into the *Challenger* space shuttle catastrophe of 1986. In front of the cameras and using just a glass of cold water and building materials, Feynman demonstrated that the failure of the *Challenger* space shuttle was due to fatigue in materials at low temperature, a thesis that was finally accepted.

In later life, Feynman was diagnosed as having two forms of cancer. These proved incurable and Richard Feynman died on 15 February 1988.

unify the three forces won the right to the name Grand Unified Theory (GUT) – although, as yet, gravity has remained beyond its scope.

Steven Weinberg, one of the three minds behind the Grand Unified Theory.

The gauge group around which the elementary particles of the Standard Model are interchanged symmetrically is $U(1) \times SU(2) \times SU(3)$. The standard model describes the interactions between the different particles that make up material. But there is nothing straightforward in the universe, and the Standard Model is no exception. Many physicists don't find it completely satisfactory, and some persistent mysteries (we might ask, for instance, why is there so much material at hand but little anti-material) have caused questions to be raised as to whether there might be some larger symmetry encompassing the Standard Model's which could help to clear up some inaccuracies. That's how supersymmetry was born.

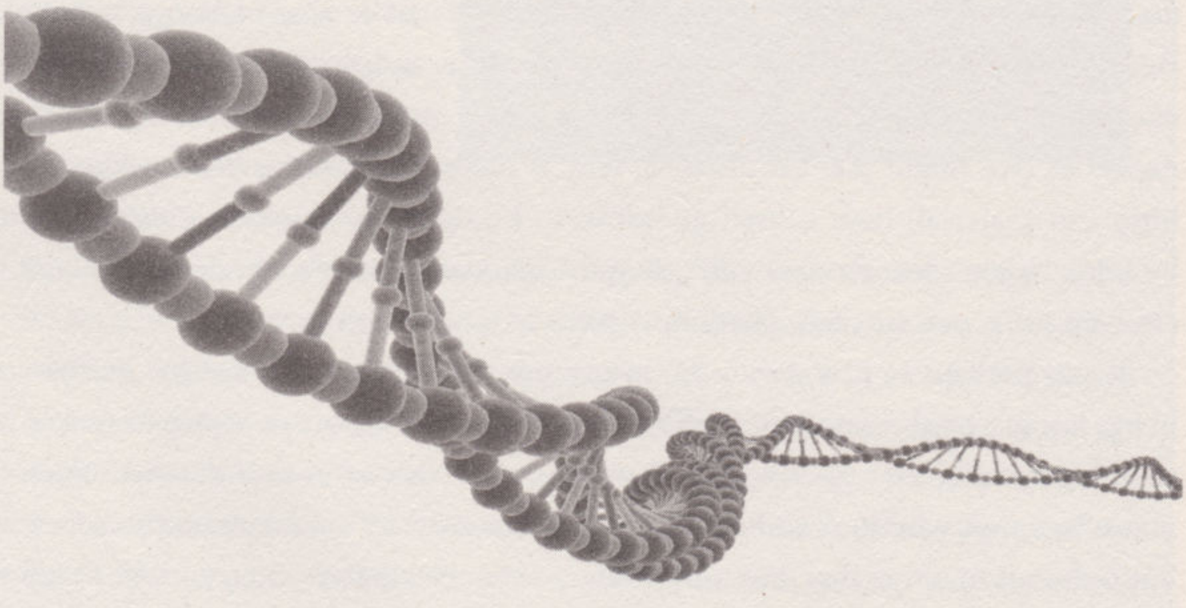
It was perhaps to play down the magnitude of the theory that it has ended up being known by the acronym SUSY (SUperSYmmetry), which seems to makes it sound cosier and less imposing. The foundation on which it rests is the incompleteness of symmetry in the Standard Model. It is true that the model unites three forces, but it does it in a way that does not please everyone. Supersymmetry posits that for every particle possessing spin, there is a symmetrical particle, called a 'sparticle'. When particle accelerators manage to reproduce conditions closer and closer to those of the Big Bang, the scientists expect to see sparticles appearing.

The most that has been achieved so far is to harmonise three basic forces in one single theory. When attempts have been made to include gravity with the others, they have failed. A neat little comparison is made by some experts who say that the gravitational field and the Standard Model's field are like children who play together but never quite manage to make friends. Attempts are made over and over again, but, there's no way they can be got together. We are searching for a "Theory of Everything" to include the gravitational field alongside the others. It may turn out to be a string theory (there are several of them) or Antony Garrett Lisi's theory which has group E_8 as its protagonist. Perhaps supersymmetry forms part of that gigantic edifice. A great deal of GUTs and lots of brain cells will be needed to build it.

A short trip through biology and chemistry

A journey through this world can be as long as one wants, as there are many reasons to stop and enjoy the displays of symmetry in nature, as many as there are books, magazines and websites devoted to the topic.

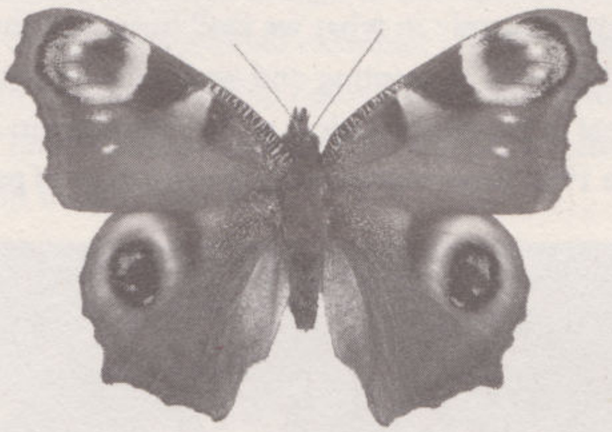
The most spectacular appearance of symmetry in recent times has perhaps been the DNA double helix, a structure popularised in general culture by James Watson (b. 1928) and Francis Crick (1916-2004), which has become an icon of human progress.



The DNA double helix.

That said, in the field of biology there are hundreds of examples of processes, patterns and forms in which symmetry leaves its mark, very often surreptitiously. Here, we shall give just a few examples. Take the narwhal, which normally has

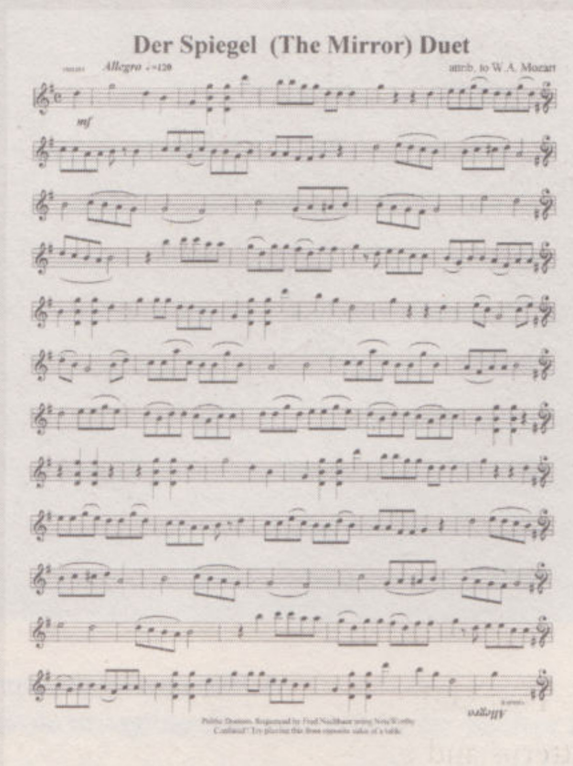
just one single tooth (it's not a horn) which displays a characteristic spiral. A few individuals, however, have two. When they do, both are helices with left-handed symmetry. As for the surface of the adenovirus, a common virus, it is icosahedral and admits simple symmetry. The horns of the wild mouflon (a species of sheep) follow a helicoidal symmetry and are enantiomorphic to each other. Bees could be considered illustrious geometricians. Attempts are currently being made to find the mechanism that impels them to construct their geometric honeycombs, which are formed from perfect hexagonal cells.



A peacock butterfly, a perfect example of bilateral symmetry?

SYMMETRICAL MUSIC

The issue of whether symmetry has had much or little influence on music could lead to a long, drawn-out discussion, but here we shall just mention one curious fact. Composers such as Bach, in *The Art of Fugue*, and Mozart created pieces of music that clearly showed symmetry. A good example, accessible to all, is the Mozart score shown right. It is played in unison by two violins, one musician reading from left to right while the other reads it in reverse.

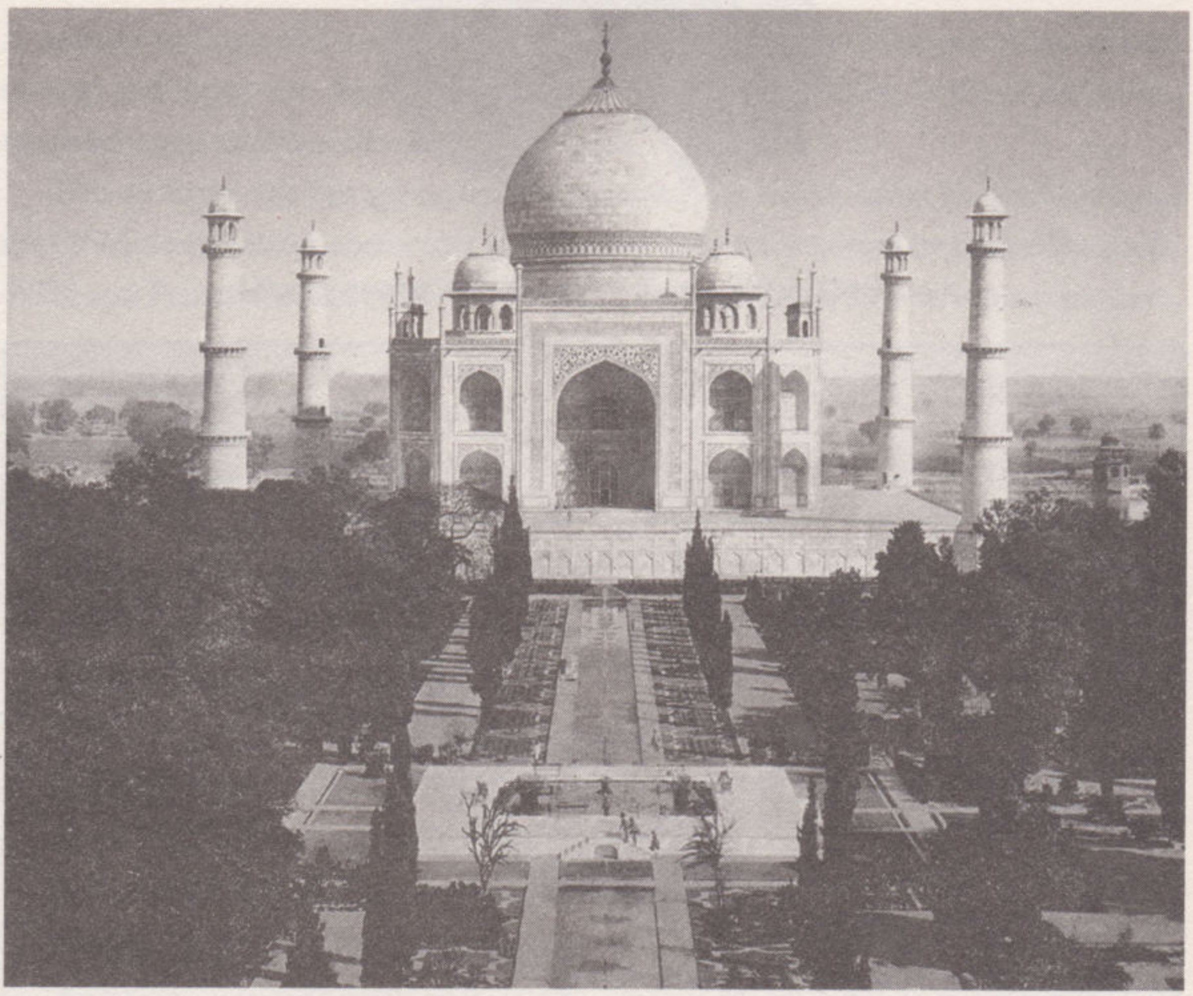


Score of the Mozart composition entitled Der Spiegel (The Mirror).

In the field of chemistry, chirality is a frequent visitor. Molecules have a symmetry dependent on the location of their atoms; dependent on that chirality, they have specific optical properties as well as specific properties of other types. As is well-known, only molecules of left-handed chirality make up the human body, and it would be reckless to ignore such a fact.

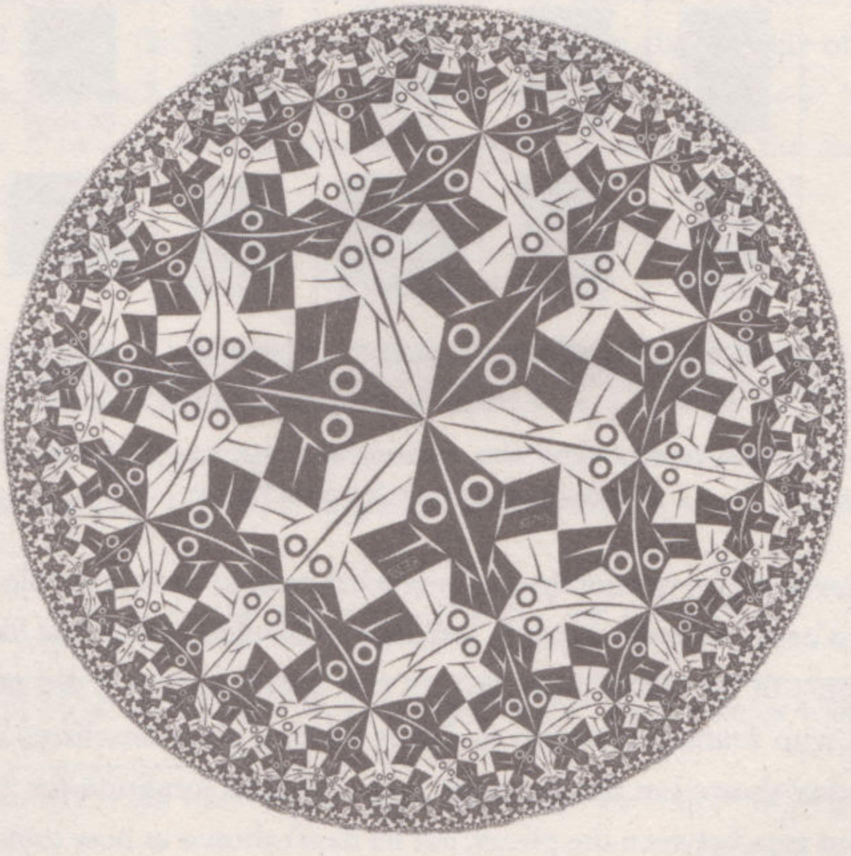
Escher and other works of art

Symmetry has an important role in what we shall lump together as simply art, but is most obvious in the fields of painting and sculpture. Examples range from the ceiling roses and arches in Gothic cathedrals to the Atomium in Brussels, a 102m tall structure built for the 1958 Expo representing the structure of part of an iron crystal.



A 19th-century photograph of the Taj Mahal, a prototype of architectural symmetry.

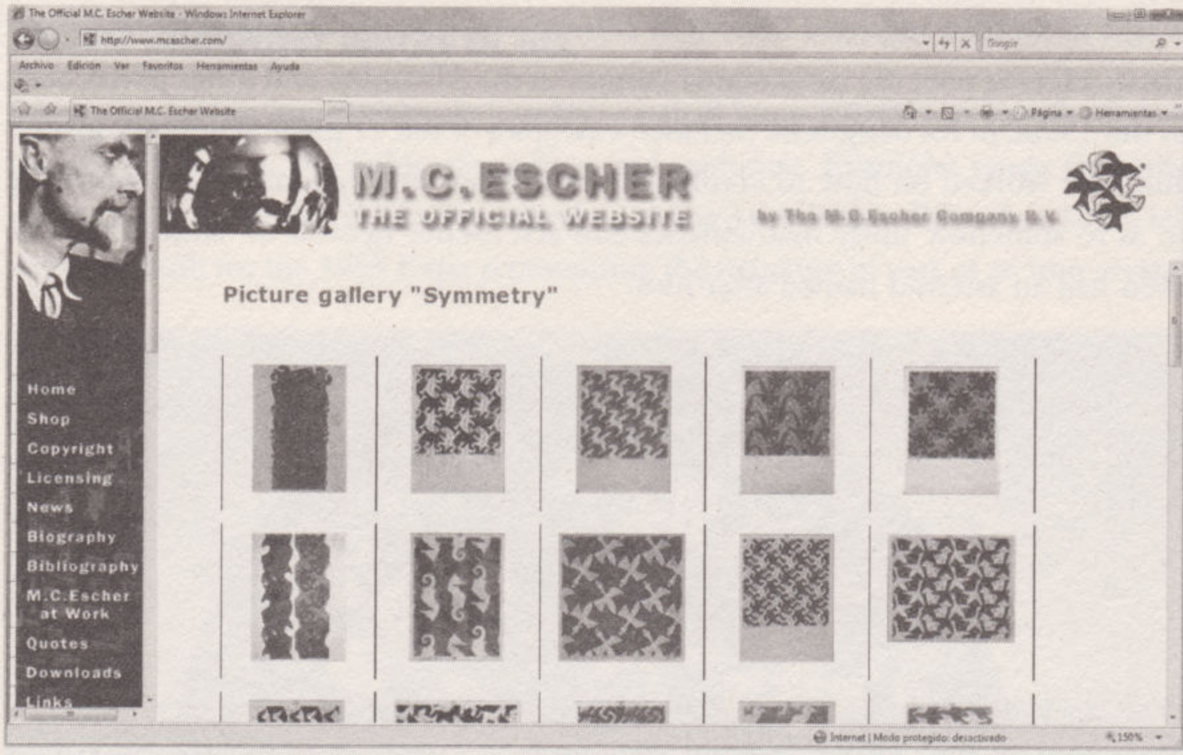
We shall not linger in this field as it is covered elsewhere in great detail. But we will make a brief mention of an artist whose always surprising, and sometimes disquieting, works reflect an astonishing vision of symmetry. Maurits Cornelis Escher (1898–1972), was a Dutch graphic artist who applied his artistic skills to the study of symmetry, tessellations, polyhedrons, impossible constructions and other works with an unmistakable mathematical vision running through them. Escher is said to be have regarded himself as closer to mathematicians than to his fellow artists. Having seen the ornamental designs in the Alhambra and other places back in the 1920s, in his designs he included examples of the 17 plane symmetry groups and later worked on how to combine colour, shape and symmetry geometrically so as to somehow unify mathematics and art. Escher became so famous that he even had an asteroid named after him.



One of the impossible symmetries conceived by M.C. Escher.

In addition to the design reproduced above, other famous works by Escher are *Drawing Hands*, which is a self-referential, symmetrical image of the artist's hand drawing itself; *Möbius strip II*, with its ants tirelessly running on a Möbius strip with

one single face; *Hand with Reflecting Sphere*, which reproduces the reflection in a spherical mirror; and the series of engravings called *Circle Limit,* dedicated to the mysterious hyperbolic symmetries. According to his friend H.S.M. Coxeter, this work was Escher *tour de force*: "Escher," said the mathematician, "did it by instinct; I did it by trigonometry."



Visit http://www.mcescher.com, a website dedicated to M.C. Escher for further information on the artist's work.

Escher devoted a lot of effort to studying what seem to be very simple tessellations, simple that is until one realises that to tessellate with inlets and outlets like those that appear in some of his works is not as easy as it looks. At least it did not look easy to Conway, who found a criterion to define when a piece tessellated and when it didn't. Conway showed us the necessary and sufficient condition for a plane to be filled without gaps between the pieces, but he hasn't shown us how such a condition could occur to someone. But it did occur to Escher, even though he knew nothing of Conway's findings.

Appendix

Lie Groups

Let's remember that a Lie group is a differentiable variety. In its most approachable form of presentation, the reader should imagine a differentiable surface that's smooth, with no folds or strange pathologies, like the surface of a sphere, a helix or a doughnut. The Lie group's elements form a group whose operations are also differentiable.

Mathematicians provide many examples of Lie groups that we could call 'normal'. One very simple one is the circumference, which we will designate radius 1 because every other group of radius r is isomorphic to it. Let's think of the subset of the complex numbers $\mathbb{C}$ formed by those of module 1, that is, of length 1, which fulfil $|z| = 1$ (let's not forget that if $z = x + iy$, the module of z, which is usually indicated with the graphics $|z|$, is the number $|z| = \sqrt{x^2+y^2}$).

Let's use S^1 to refer to such a set of points, which is simply the circumference of unitary radius,

$$S^1 = \{z \mid z \in \mathbb{C}, |z| = 1\}.$$

It turns out that S^1 is also a group with the multiplication operation. In fact, the operation is internal, as, for any complex numbers, $|z_1| \cdot |z_2| = |z_1 \cdot z_2|$ and, therefore, if the two complexes z_1 and z_2 are of module or length 1, their product is too.

Let's prove that $|z_1| \cdot |z_2| = |z_1 \cdot z_2|$, and let's write for it $z_1 = a + ib$ and $z_2 = c + id$.

$$|z_1|\cdot|z_2| = \sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2} + \sqrt{(a^2+b^2)(c^2+d^2)} =$$

$$= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2-2abcd+2abcd} =$$

$$= \sqrt{(ac-bd)^2+(ad+bc)^2} = |(ac-bd)+(ad+bc)i| = |(a+ib)(c+id)| = |z_1\cdot z_2|.$$

If $z = x + iy$, it follows, by doing a few arithmetic calculations, that

$$1/z = z^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

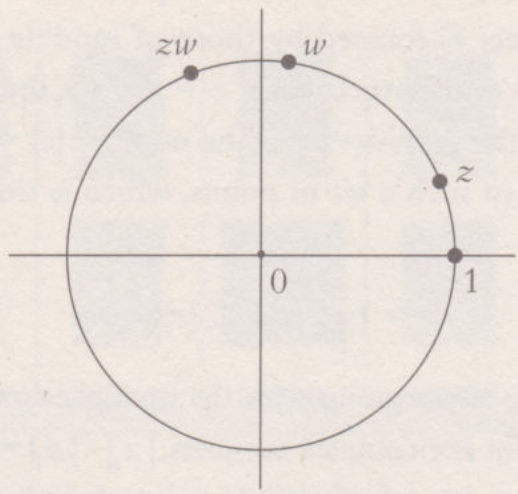
and it's not difficult to prove that

$$z^{-1} = \frac{1}{z} \in S^1.$$

If we make use of elementary arithmetic, we find that

$$\left|z^{-1}\right| = \left|\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right| = \sqrt{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}} = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{1}} = 1.$$

The groups' other conditions are easy to see. The neutral element is number 1 ($1 = 1 + i0$). So if the two complex numbers, z and w of module 1, are multiplied, the result is another complex number, zw, also of module 1.



When two points of the circumference are multiplied, they give another point of the circumference. We now have a simple Lie group example, to which we shall add a few more.

To make it easier to discuss Lie groups, it is necessary to use matrix algebra. To clarify, a matrix is a simple, rectangular, or sometimes square, table whose numbers can be added or multiplied.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

with $c_{ij} = a_{ij} + b_{ij}$ and

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix}$$

with $d_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + ... + a_{n1}\, b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

There are sets of matrices that form a group – sometimes they're Abelian, other times they aren't – with the operation + or with the operation ·. In the first case – the addition operation – it is easy to show that the neutral element is the matrix "zero".

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

And the matrix opposite the A is the one called $-A$.

$$A + (-A) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = -A + A = 0.$$

To calculate the inverses it is enough to know that the identity matrix for the operation · is the matrix which has 1 on the diagonal and 0 in the other positions.

THE CARTAN CLAN

The Cartans, father and son, were outstanding French mathematicians who both devoted a great deal of effort to group theory. Élie Cartan (1869–1951) was from a very humble background and is the perfect example of a self-made man. He came out of abject poverty but was to die as an emeritus professor, respected by all, the recipient of honorary doctorates from numerous universities and with a growing reputation and prestige within his profession. His whole life was dedicated to studying differential geometry, in particular Lie groups and Lie algebra. He based his early efforts on the fledgling ideas contained in the work of Wilhelm Killing (1847–1923), and was to make Lie groups into material that attracted an exceedingly high level of interest. In 1913 he conceived the idea of *spinors*, a mathematical concept on vector generalisation that was to play an important role in physics.

During World War II, the Nazis beheaded one of his sons who was accused of belonging to the Resistance, which, together with the premature death of another son, caused Élie great sorrow, filling him with despair. Fortunately, his remaining son Henri was to be a great joy to him.

Henri Cartan (1904–2008) inherited his father's gifts. He devoted himself to mathematics of his own free will, he was not forced into it by his father and they even wrote an occasional article together. He worked at a very high level of science, becoming a peerless specialist in complex functions, which in turn led him to consider varieties and the field of homological algebra, a strongly developing abstract field of algebra. Despite the personal tragedy of seeing the Nazis murder his brother, after the war Cartan acted as a bridge between French and German mathematicians. This humanitarian attitude would never leave him. His struggles against totalitarian regimes to gain freedom for professional colleagues like Plyusch and Massera are legendary.

Élie Cartan photographed in 1903.

At the end of the 1950s, he began a series of visits to the United States, where he met Samuel Eilenberg (1913–1998). Their friendship spawned the joint creation and publica-

tion of *Homological Algebra*. Rarely has something so difficult left such a profound imprint and originated so much research and new work.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d, "\partial_{n+1}"'] & B_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d, "\partial'_{n+1}"'] & C_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"'] & 0 \\
0 \arrow[r] & A_{n} \arrow[r, "\alpha_{n}"] \arrow[d, "\partial_{n}"'] & B_{n} \arrow[r, "\beta_{n}"] \arrow[d, "\partial'_{n}"'] & C_{n} \arrow[r] \arrow[d, "\partial''_{n}"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

A typical homological algebra scheme. The Cartan and Eilenberg's text is so abstruse, dense and intimidating – but beautiful – that the experts call it "the diplodocus".

The list of honours and honorary doctorates bestowed on Henri in his old age exceeds even his father's. As an anecdote, it should also be mentioned that Henri Cartan played a leading role in the conceptualisation and development of the work of Nicolas Bourbaki, the name under which a collective of mathematicians published their work.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

As for the components in the rows and columns of the inverse matrix, A^{-1}, we have to resolve a system of n^2 linear equations whose solutions, b_{ij}, depend on the initial conditions:

$$c_{ij} = a_{i1}b_{1j} + a_{12}b_{2j} + \ldots + a_{1n}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}, \text{ with } c_{ij} = 1 \text{ if } i = j, c_{ij} = 0 \text{ if } i \neq j.$$

Sometimes the system can be resolved, sometimes it can't.

Matrices were introduced by Cayley, after several previous attempts by other scientists and, as a help in writing, they are invaluable for algebraists.

The virtue of matrices is that they will allow us to discuss several Lie groups with ease. For example, the matrices of real numbers:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad\text{-}bc \neq 0$, that is, with a non-null determinant, is a Lie group. It is called GL_2 ($\mathbb{R}$), on account of the size of the matrices. With n rows and columns it changes into the general linear group $GL_n(\mathbb{R})$. A subgroup of this is $SL_n(\mathbb{R})$, the special linear group, which is distinguished by having matrices of determinant 1.

One important group is $O_n(\mathbb{R})$, the group of orthogonal real numbers (they have a determinant equal to $\pm$ 1); a subgroup of this is $SO_n(\mathbb{R})$, which have determinant 1 and comprise the rotations.

With SO_2 ($\mathbb{R}$) we get matrices of the

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \alpha \in$$

descriptive type of the turns on the plane.

A Lie group which is very important in physics is the unitary group denominated $U(n)$, consisting of unitary groups of n rows and n columns in the field of complex numbers. To make it comprehensible, imagine the space built with complexes instead

of real numbers, and provided with a metric, established through a scalar product; the unitary matrices are those that describe isometries, that is, transformations that preserve the distance. The matrices of determinant 1 form a subgroup, $SU(n)$, the special unitary subgroup.

It is not difficult to show, for instance, that

$$S^1 \cong U(1) \cong SO_2(\mathbb{R}).$$

Precisely, the product group $U(1) \times SU(2) \times SU(3)$ is used in particle physics as a *gauge* symmetry group (sometimes called measurement symmetry) for the standard model.

Linked to this group is the Lie group called E_8, an infinite and sporadic Lie group of dimension 278. Despite its enormous size – each element is a square matrix with 453,060 rows and columns – it is currently being researched as it is a good candidate for building a *Theory for Everything*.

The Lorentz group, a Lie group which is widely used in higher physics, is a subgroup of the Poincaré group. The largest group is the group of all the isometries of relativist space-time; the smallest, which is hexadimensional, is that of those isometries that leave the origin fixed.

There are many other Lie groups too, each of them with its name and characteristics; the best thing to do is to stop here and explain each group when needed.

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Index

Creating the Perfect Balance

Symmetry in maths

The study of symmetry is an attempt to understand beauty better – and enjoy it more. Ideals of beauty are, to a great extent, based on notions of proportionality and equilibrium. Furthermore, symmetry constitutes a fundamental concept in the study of the world around us. There are many symmetries, ranging from the fictional Alice and her journey through the looking glass, to the fascinating world of group theory. The intriguing field of symmetry offers a deeper insight into one of the fundamental threads in the tapestry of contemporary mathematics.